

# Evolution operators and quantum chains

Sergey M. SERGEEV

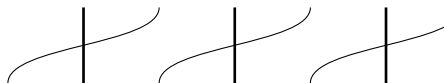
Department of Theoretical Physics, RSPSE  
Department of Mathematics, MSI  
The Australian National University

Statistical Mechanics Meeting, Sydney, December 11 - 12, 2006

## QUANTUM CHAIN

[algebra of observables, quantum Lax operators, YBE, etc. ]

- ▶ Transfer matrix
- ▶ Hamiltonian as a local operator in algebra of observables
- ▶ Bethe Ansatz and application to physics
- ▶ Evolution operator
- ▶ Hamiltonian as the logarithm of evolution operator
- ▶ Bethe Ansatz and application to physics



## Attractive lattice Bose gas with XXZ-type impurities

Algebra of observables is the set  $\mathbf{a}_{j,n}, \mathbf{a}_{j,n}^\dagger, \mathbf{n}_{j,n}$  [ $j = 1, 2, 3, n = 1, \dots, M$ ] of  $q$ -oscillators,

$$[\mathbf{a}, \mathbf{a}^\dagger] = (1 - q^2)q^{2n}, \quad [\mathbf{a}, \mathbf{n}] = -\mathbf{a}, \quad [\mathbf{a}^\dagger, \mathbf{n}] = \mathbf{a}^\dagger$$

Two types of quantum Lax operators and their transfer matrix:

$$M_n(u) = \begin{pmatrix} q^{n_{3,n}} & u\mathbf{a}_{3,n}^\dagger \\ -q^{-1}\mathbf{a}_{3,n} & uq^{n_{3,n}} \end{pmatrix} \quad \text{/Bose gas type/}$$

$$L_n(u) = \begin{pmatrix} uq^{n_{1,n}} - e^{i\varepsilon}q^{n_{2,n}} & -ue^{i\varepsilon}\mathbf{a}_{1,n}\mathbf{a}_{2,n}^\dagger \\ -q^{-1}\mathbf{a}_{2,n}\mathbf{a}_{1,n}^\dagger & uq^{n_{2,n}} - e^{i\varepsilon}q^{n_{1,n}} \end{pmatrix} \quad \text{/XXZ type /}$$

$$t(u) = \text{Tr} \left( M_1(u)L_1(u)M_2(u)L_2(u) \cdots M_N(u)L_N(u) \right)$$

Conserving charges: local spins and the occupation number

$$s_n = \frac{1}{2}(\mathbf{n}_{1,n} + \mathbf{n}_{2,n}) \quad \text{/centers of } L_n/ \quad \text{and} \quad \mathcal{K} = \sum_n (\mathbf{n}_{2,n} + \mathbf{n}_{3,n})$$

## Evolution operator

Definition of  $\mathcal{U}$ :

$$\mathcal{U}M_n(u)L_n(u) = L_n(u)M_{n+1}(u)\mathcal{U} \quad \Rightarrow \quad \mathcal{U}t(u) = t(u)\mathcal{U} .$$

Corollary:  $\mathcal{U}|0\rangle = |0\rangle$  and

$$\begin{aligned} \mathcal{U}q^{n_2,n}a_{1,n}^\dagger\mathcal{U}^{-1} &= q^{n_3,n}a_{1,n}^\dagger - e^{i\varepsilon}q^{n_1,n}a_{2,n}^\dagger a_{3,n} \\ \mathcal{U}a_{2,n}^\dagger\mathcal{U}^{-1} &= a_{1,n}^\dagger a_{3,n}^\dagger + qe^{i\varepsilon}q^{n_1,n+n_3,n}a_{2,n}^\dagger \\ \mathcal{U}q^{n_2,n+1}a_{3,n}^\dagger\mathcal{U}^{-1} &= q^{n_1,n+1}a_{3,n+1}^\dagger - e^{i\varepsilon}q^{n_3,n+1}a_{1,n+1}^\dagger a_{2,n+1}^\dagger \end{aligned}$$

Test: choose a state and start up the clock.

$$|\psi_1\rangle = a_{2,0}^\dagger|0\rangle, \quad |\psi_0\rangle \otimes |a_n\rangle = a_{1,0}^\dagger a_{3,n}^\dagger|0\rangle,$$

These states correspond to  $s_0 = \frac{1}{2}$ , all other  $s_n = 0$ ,  $\mathcal{K} = 1$ . Then

$$\mathcal{U}|\psi_1\rangle = e^{i\varepsilon}q|\psi_1\rangle + |\psi_0\rangle \otimes |a_0\rangle, \quad \mathcal{U}^2|\psi_1\rangle = \dots,$$

and so on ...

$q = e^{-\gamma}$ ,  $T \gg 1$ ,  $\gamma T \ll 1$ ,  $\varepsilon T \sim$  some integer multiply of  $2\pi$

$$\mathcal{U}^T |\psi_1\rangle = e^{(i\varepsilon - \gamma)T} |\psi_1\rangle + |\psi_0\rangle \otimes \left( \sum_{n=0}^{T-1} e^{(i\varepsilon - \gamma)(T-n-1)} |a_n\rangle \right)$$

- ▶ Decay of excited state  $|\psi_1\rangle$  with the energy  $\varepsilon$  and the width  $\gamma$
- ▶ Spontaneous radiation of the right-moving wave package with the average energy and momentum  $\varepsilon$
- ▶ Radiated packages are photons, sites  $L_n$  with  $s_n = 1/2$  are two-states atoms, sites  $L_n$  with  $s_n = 0$  are transparent.

Bethe Ansatz and spectrum of  $\mathcal{U}$ :

$$u_k^N \prod_n \frac{u_k - q^{2s_n} e^{i\varepsilon}}{q^{2s_n} u_k - e^{i\varepsilon}} = \prod_{k' \neq k} \frac{q^{-1} u_k - q u_{k'}}{q u_k - q^{-1} u_{k'}}, \quad k, k' = 1, \dots, \mathcal{K},$$

$$u_k = e^{ip_k}, \quad \text{Re}(p_k) \geq 0, \quad \mathcal{U} = \exp \left\{ i \sum_{k=1}^{\mathcal{K}} p_k \right\}.$$

$q = e^{-\gamma}$ ,  $T \gg 1$ ,  $\gamma T \ll 1$ ,  $\varepsilon T \sim$  some integer multiply of  $2\pi$

$$\mathcal{U}^T |\psi_1\rangle = e^{(i\varepsilon - \gamma)T} |\psi_1\rangle + |\psi_0\rangle \otimes \left( \sum_{n=0}^{T-1} e^{(i\varepsilon - \gamma)(T-n-1)} |a_n\rangle \right)$$

- ▶ Decay of excited state  $|\psi_1\rangle$  with the energy  $\varepsilon$  and the width  $\gamma$
- ▶ Spontaneous radiation of the right-moving wave package with the average energy and momentum  $\varepsilon$
- ▶ Radiated packages are photons, sites  $L_n$  with  $s_n = 1/2$  are two-states atoms, sites  $L_n$  with  $s_n = 0$  are transparent.

Bethe Ansatz and spectrum of  $\mathcal{U}$ :

$$u_k^N \prod_n \frac{u_k - q^{2s_n} e^{i\varepsilon}}{q^{2s_n} u_k - e^{i\varepsilon}} = \prod_{k' \neq k} \frac{q^{-1} u_k - q u_{k'}}{q u_k - q^{-1} u_{k'}}, \quad k, k' = 1, \dots, \mathcal{K},$$

$$u_k = e^{ip_k}, \quad \text{Re}(p_k) \geq 0, \quad \mathcal{U} = \exp \left\{ i \sum_{k=1}^{\mathcal{K}} p_k \right\}.$$

$$q = e^{-\gamma}, \gamma \ll \varepsilon \ll 1, \text{Re}(p) > 0$$

Photonic counterpart of eigenstates, real  $p_k > 0$ :

$$|\Psi\rangle \sim \prod_k a_{p_k}^\dagger |0\rangle, \quad a_p^\dagger = \sum e^{-ipn} a_{3,n}^\dagger$$

Strings of Bethe Ansatz,  $p_1 = p - i\gamma$  and  $p_2 = p + i\gamma$ ,  $p > 0$ :

$$|\Psi\rangle \sim \sum_{n,n'} e^{-ip(n+n') - \gamma|n-n'|} a_{3,n}^\dagger a_{3,n'}^\dagger |0\rangle \sim a_p^{\dagger 2} |0\rangle$$

Ground state:  $\mathcal{K}$ -string condensate  $p_k = p_0 + i(\mathcal{K} + 1 - 2k)\gamma$ ,

$$|\Psi_0\rangle \sim a_{p_0}^{\dagger \mathcal{K}} |0\rangle, \quad E_0 = \mathcal{K}p_0 \sim 1/N$$

## Optical wave band $p \sim \varepsilon$

The low energy and is the radio wave band.

Next topics of interest: the optical wave band  $p \sim \varepsilon$ .

$$\#(s_n = 1/2) = N' , \quad \#(s_n = 0) = N - N' .$$

Choose  $Im(p_k) = 0$  near  $p_k \sim \varepsilon$ , assume dense distribution  $\varrho(p_k)^{-1} = N(p_{k+1} - p_k)$ , then

$$1 - 2\pi\varrho(p) = \int S_2(p-p')\varrho(p')dp' - \frac{N'}{N}S_1(p-\varepsilon), \quad S_n(p) = \frac{2n\gamma}{p^2 + (n\gamma)^2}$$

Solution:

$$\varrho(p) \simeq const + \frac{N'}{4\gamma N} \frac{1}{\cosh \frac{\pi(p-\varepsilon)}{2\gamma}}$$

what is resonance peak over a white noise.

In the full-load regime the laser does work.

THANK YOU