Tetrahedron Equations and Integrability

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Outline

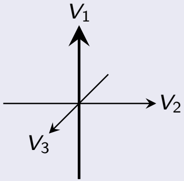
1. General framework of the three-dimensional quantum integrability

2. Particular example: q-oscillator model

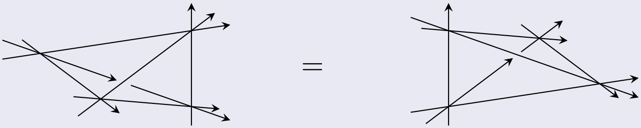
Vertex operators and Tetrahedron equations

3D vertex

- $R_{V_1,V_2,V_3} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$
- Spaces V_1, V_2, ... may be equipped by independent spectral parameters
- Spaces V_1, V_2, ... may have different structure (e.g dimension)



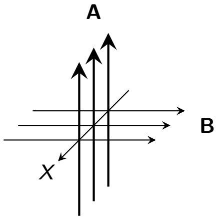
$$R_{V_1,V_2,V_3}R_{V_1,V_4,V_5}R_{V_2,V_4,V_6}R_{V_3,V_5,V_6} = R_{V_3,V_5,V_6}R_{V_2,V_4,V_6}R_{V_1,V_4,V_5}R_{V_1,V_2,V_3}$$



Chain of 3D vertices

$$R_{\mathbf{A},\mathbf{B},X} = R_{A_1,B_1,X} R_{A_2,B_2,X} \cdots R_{A_n,B_n,X}$$

$$\mathbf{A} = \bigotimes_i A_i \,, \quad \mathbf{B} = \bigotimes_i B_i$$
$$R_{\mathbf{A},\mathbf{B}}[X] = \mathrm{Trace}_X R_{\mathbf{A},\mathbf{B},X}$$



Set of the Tetrahedron equations

$$\prod_i^{\curvearrowright} R_{A_i,B_i,X} R_{A_i,C_i,Y} R_{B_i,C_i,Z} \cdot R_{X,Y,Z} = R_{X,Y,Z} \cdot \prod_i^{\curvearrowright} R_{B_i,C_i,Z} R_{A_i,C_i,Y} R_{A_i,B_i,X}$$

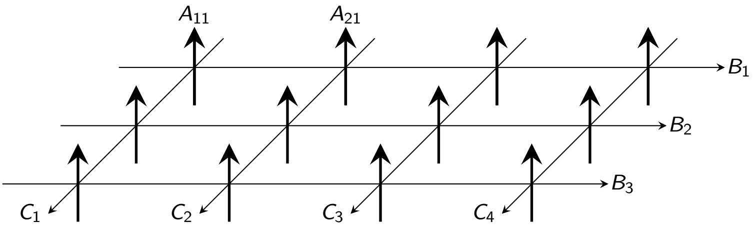
yields the Yang-Baxter Equation in the tensor powers $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}$

$$R_{\mathbf{A},\mathbf{B}}[X] R_{\mathbf{A},\mathbf{C}}[Y] R_{\mathbf{B},\mathbf{C}}[Z] = R_{\mathbf{B},\mathbf{C}}[Z] R_{\mathbf{A},\mathbf{C}}[Y] R_{\mathbf{A},\mathbf{B}}[X]$$

Layer-to-Layer transfer matrix

$$\text{Layer-to-Layer TM:} \quad T_{\mathbf{A}}[B,C] = \operatorname{Trace}_{\mathbf{B},\mathbf{C}} \overset{\curvearrowright}{\prod_i} \overset{\curvearrowright}{\prod_k} R_{A_{ik},B_i,C_k}$$

$$\mathbf{A} = \bigotimes_{i,k} A_{ik} \,, \quad \mathbf{B} = \bigotimes_i B_i \,, \quad \mathbf{C} = \bigotimes_k C_k$$

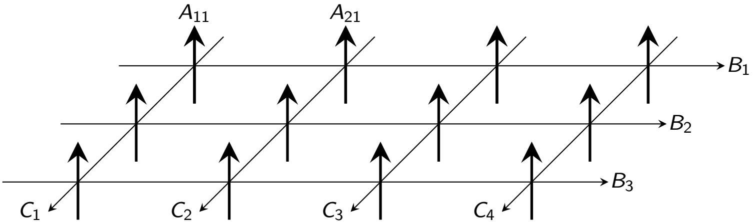


$$T_{\mathbf{A}}[B,C]\; T_{\mathbf{A}}[B',C'] \;=\; T_{\mathbf{A}}[B',C']\; T_{\mathbf{A}}[B,C]$$

Summary

We have seen: the layer corresponds to a two-dimensional quantum lattice – higher-dimensional generalization of a quantum chain.

The way back to 2D: Collection of the nodes of square quantum lattice along one direction is a site of effective chain [compactification]. The number n of the nodes in the chosen direction is related to the rank of effective symmetry group $[U(\widehat{sl}_n)]$.



Spaces in q-oscillator model

Vector spaces V and $\mathcal{F}$:

- Two-dimensional vector space V

$$V \, [\equiv \mathbb{C}^2] \;=\; \{|0\rangle, |1\rangle\} \;\text{ or }\; \{|\uparrow\rangle, |\downarrow\rangle\} \;\text{ or }\; \{|\mathit{white}\rangle, |\mathit{black}\rangle\}$$

- and the Fock space $\mathcal{F}$,

$$\mathbf{B}\mathbf{B}^\dagger = 1 - q^{2+2\mathbf{N}} \;, \quad \mathbf{B}^\dagger\mathbf{B} = 1 - q^{2\mathbf{N}}$$

$$\mathcal{F} \;:\; \text{Spectrum}(\mathbf{N}) = 0, 1, 2, \ldots,$$

Fock space $\mathcal{F}_j$ is equipped by a pair of auxiliary parameters λ_j, μ_j.

Vertex operators in the q-oscillator model

$L^{(q)} \in \mathrm{End}(V \otimes V \otimes \mathcal{F})$

In the basis of $V \otimes V = \{|0,0\rangle, |1,0\rangle, |0,1\rangle, |1,1\rangle\}$

$$L^{(q)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda q^{\mathbf{N}} & \nu \mathbf{B}^{\dagger} & 0 \\ 0 & \nu \mathbf{B} & \mu q^{\mathbf{N}} & 0 \\ 0 & 0 & 0 & \nu^2 \end{pmatrix}, \quad \nu^2 = -q^{-1}\lambda\mu;$$

λ, μ equip $\mathcal{F}$.

$R^{(q)} \in \mathrm{End}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3)$

Expression for the matrix elements $\langle n_1, n_2, n_3 | R | m_1, m_2, m_3 \rangle$ is known.

Tetrahedrons

$$L^{(q)}_{\alpha,\beta;f} \in \mathrm{End}(V_\alpha \otimes V_\beta \otimes \mathcal{F}_f),\ R^{(q)}_{1,2,3} \in \mathrm{End}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3).$$

Three types of tetrahedron equations:

$$L^{(q)}_{\alpha,\beta;1}\ L^{(q)}_{\alpha,\gamma;2}\ L^{(q)}_{\beta,\gamma;3}\ R^{(q)}_{1,2,3} = R^{(q)}_{1,2,3}\ L^{(q)}_{\beta,\gamma;3}\ L^{(q)}_{\alpha,\gamma;2}\ L^{(q)}_{\alpha,\beta;1}$$

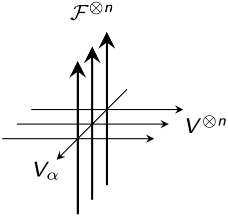
$$R^{(q)}_{1,2,3}\ R^{(q)}_{1,4,5}\ R^{(q)}_{2,4,6}\ R^{(q)}_{3,5,6} = R^{(q)}_{3,5,6}\ R^{(q)}_{2,4,6}\ R^{(q)}_{1,4,5}\ R^{(q)}_{1,2,3}$$

$$L^{(-q)}_{\alpha,\alpha';0}\ L^{(-q)}_{\beta,\beta';0}\ L^{(q)}_{\alpha,\beta;f}\ L^{(q)}_{\alpha',\beta';f} = L^{(q)}_{\alpha',\beta';f}\ L^{(q)}_{\alpha,\beta;f}\ L^{(-q)}_{\beta,\beta';0}\ L^{(-q)}_{\alpha,\alpha';0}$$

Lax operators and R-matrices of $\mathcal{U}_q(\widehat{sl}_n)$

$$\mathfrak{L}_{\boldsymbol{\beta};\boldsymbol{\mathcal{F}}}(u) \;=\; \operatorname*{Trace}_{V_\alpha}\left(D_\alpha(u) \overset{\curvearrowright}{\prod_i} L^{(q)}_{\alpha,\beta_i;\mathcal{F}_i} \right), \quad D(u) = \begin{pmatrix} 1 & 0 \\ 0 & -u \end{pmatrix}$$

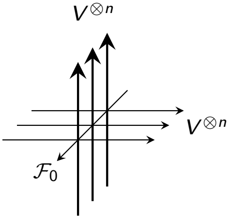
Invariant subspaces:
$V_{\boldsymbol{\beta}} = V^{\otimes n} \ni |k_1, k_2, ..., k_n\rangle$, $k_i = 0, 1$. Invariant subspace corresponds to fixed value of
$\ell = k_1 + k_2 + \cdots + k_n$.
Invariant of $\boldsymbol{\mathcal{F}} = \mathcal{F}^{\otimes n}$ is
$J \;=\; \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_n$.



$$V^{\otimes n} = \bigoplus_{\ell=0}^{n} \pi_{\omega_\ell} \quad \text{[antisymmetric tensors of } sl_n\text{]}$$

$$\mathcal{F}^{\otimes n} = \bigoplus_{J=0}^{\infty} \pi_{J\omega_1} \quad \text{[symmetric tensors of } sl_n\text{]}$$

Fundamental R-matrices of $\mathcal{U}_q(\widehat{sl}_n)$



$$\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}(u/u') = \operatorname*{Trace}_{\mathcal{F}_0}\left((u/u')^{-\mathbf{N}_0} \prod_i^{\curvearrowright} L^{(-q)}_{\beta_i,\beta'_i;\mathcal{F}_0}\right) = \sum_{\ell,\ell'=0}^{n} \lambda_0^{\ell}\mu_0^{\ell'} R_{\omega_\ell,\omega_{\ell'}}(u/u')$$

Transfer matrices

$$T[u,v] = \operatorname*{Trace}_{V_{\boldsymbol{\alpha}},V_{\boldsymbol{\beta}}}\left(D_{\boldsymbol{\beta}}(v) D_{\boldsymbol{\alpha}}(u) \prod_k^{\curvearrowright} \prod_i^{\curvearrowright} L_{\alpha_k \beta_i ; \mathcal{F}_{ki}} \right)$$

$$D_{\boldsymbol{\alpha}} = \prod_{k=1}^{m} D_{\alpha_k}(u) \,, \quad D_{\boldsymbol{\beta}} = \prod_{i=1}^{n} D_{\beta_i}(v)$$

Rank-size $n \Leftrightarrow m$ duality:

$$T[u,v] = \sum_{i=0}^{n} \sum_{k=0}^{m} v^i \; u^k \; T_{i,k} \equiv \sum_{i=0}^{n} v^i \, T_{\omega_i}^{sl_n}[u] \equiv \sum_{k=0}^{m} u^k \, T_{\omega_k}^{sl_m}[v]$$

Spectral equations

Spectral equations in the terms of complete set of transfer-matrices $T^{sl_n}_{\omega_i}(u)$:

$$\sum_{i=0}^{n} Q(q^{2i}u)v^i T^{sl_n}_{\omega_i}[(-q)^i u] = 0$$

"For some fixed values of v function $Q(u)$ is a polynomial" $\Leftrightarrow$ NBAE.

Dual Bethe-Ansatz:

$$\sum_{k=0}^{m} T^{sl_m}_{\omega_k}[(-q)^k v] u^k \widetilde{Q}(q^{2k}v) = 0 \,.$$

2D versus 3D

Beyond 2D: essential 3D way

- Quantum lattice may have various shapes [Kagome lattice for instance]
- Quantum lattice may have boundary structure such that compactification is not possible
- Square quantum lattice with the sizes $m \times n$: nested Bethe Ansatz technique implies $m \to \infty$ with finite n $[n/m \to 0]$. Two-dimensional methods do not describe the regime of non-singular aspect ratio n/m.

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THANK YOU

$$R_{123}\phi(u_1, w_1, u_2, w_2, u_3, w_3)R_{123}^{-1} = \phi(u_1', w_1', u_2', w_2', u_3', w_3')$$

given by

$$\begin{array}{lll} w_1' = w_2\,\Lambda_3, & w_2' = \Lambda_3^{-1}\,w_1, & w_3' = \Lambda_2^{-1}\,u_1^{-1}, \\ u_1' = \Lambda_2^{-1}\,w_3^{-1}, & u_2' = \Lambda_1^{-1}\,u_3, & u_3' = u_2\,\Lambda_1, \end{array}$$

where

$$\begin{aligned} \Lambda_1 &= u_1^{-1}u_3 + q u_1^{-1}w_1 + \varkappa_1 w_1 u_2^{-1}, \\ \Lambda_2 &= \tfrac{\varkappa_1}{\varkappa_2}u_2^{-1}w_3^{-1} + \tfrac{\varkappa_3}{\varkappa_2}u_1^{-1}w_2^{-1} + q^{-1}\tfrac{\varkappa_1\varkappa_3}{\varkappa_2}u_2^{-1}w_2^{-1}, \\ \Lambda_3 &= w_1 w_3^{-1} + q u_3 w_3^{-1} + \varkappa_3 w_2^{-1}u_3. \end{aligned}$$

q-oscillator algebra $xy = 1 - q^2 k^2\,, \quad yx = 1 - k^2$

$$R_{123}\phi(x_1, y_1, x_2, y_2, x_3, y_3)R_{123}^{-1} = \phi(x_1', y_1', x_2', y_2', x_3', y_3')$$

given by the recursive set of equations:

$$k_2'^2 = k_1^2 k_2^2 k_3^2 - (1+q^2)k_1^2 k_2^2 + k_1^2 + k_2^2 - \frac{1}{\lambda_1\mu_3} k_1 k_3 y_1 x_2 y_3 - \lambda_1\mu_3 k_1 k_3 x_1 y_2 x_3$$

$$k_1' = k_2'^{-1} k_1 k_2\,, \quad k_3' = k_2'^{-1} k_2 k_3 \quad \text{and}$$

$$x_1' = \tfrac{\lambda_2}{\lambda_3} k_2'^{-1}\left(k_3 x_1 - \tfrac{q}{\lambda_1\mu_3} k_1 x_2 y_3\right), \quad y_1' = \tfrac{\lambda_3}{\lambda_2} k_2'^{-1}\left(k_3 y_1 - \tfrac{\lambda_1\mu_3}{q} k_1 y_2 x_3\right),$$

$$x_2' = \left(x_1 x_3 + \tfrac{q^2}{\lambda_1\mu_3} k_1 k_3 x_2\right), \quad y_2' = \left(y_1 y_3 + \lambda_1\mu_3 k_1 k_3 y_2\right),$$

$$x_3 = \tfrac{\mu_2}{\mu_1} k_2'^{-1}\left(k_1 x_3 - \tfrac{q}{\lambda_1\mu_3} k_3 y_1 x_2\right), \quad y_3 = \tfrac{\mu_1}{\mu_2} k_2'^{-1}\left(k_1 y_3 - \tfrac{\lambda_1\mu_3}{q} k_3 x_1 y_2\right).$$

Weyl Algebra $uw = q^2 wu$. II

Solution of the 4-simplex equation:

$$R_{0123}\phi(u_0, w_0, u_1, w_1, u_2, w_2, u_3, w_3)R_{0123}^{-1} = \phi(u_0', w_0', u_1', w_1', u_2', w_2', u_3', w_3')$$

given by

$$\begin{array}{ll}
u_0' = u_1 w_3^{-1}, & w_0' = (w_1 + qu_3)w_3^{-1}, \\
u_1' = u_2 + u_0 w_2, & w_1' = w_0 w_2 \\
u_2' = qu_1 u_3 (w_1 + qu_3)^{-1}, & w_2' = w_1 w_3 (w_1 + qu_3)^{-1} \\
u_3' = \ qu_0^{-1} w_0 u_2, & w_3' = u_0^{-1} u_2 + w_2
\end{array}$$

$R_{123} = \mathrm{Trace}_0 R_{0123}$ satisfies the tetrahedron equation.

Two-dimensional lattice Bose Gas

$$\omega_i = e^{ik_{x,i}}, \quad \Omega_i = e^{ik_{y,i}}, \quad i = 1, 2, \ldots, p,$$

$$G_{ik}[\{\omega\}] \; = \; \frac{q^{-1}\omega_k - q\omega_i}{\omega_k - \omega_i} \; .$$

Let I_a be a length-a subsequence of $(1, 2, \ldots, p)$, I_{p-a} be the compliment subsequence such that $I_a \cup I_{p-a} = (1, 2, \ldots, p)$.

$$S_a[\{\omega\}] = \sum_{I_a} \prod_{i \in I_a} \omega_i$$

$$P_a[\{\omega, \omega^M\}] \; = \; \sum_{I_a} \left(\prod_{i \in I_a} \omega_i^M \prod_{k \in I_{p-a}} G_{k,i}[\{\omega\}] \right) \; ,$$

$$S_a[\{\Omega\}] = P_a[\{\omega, \omega^M\}], \quad S_a[\{\omega\}) = P_a[\{\Omega, \Omega^N\}]$$