q-oscillator lattice and representations of quantum groups

S. Sergeev

Department of Theoretical Physics
Research School of Physical Sciences and Engineering
Australian National University

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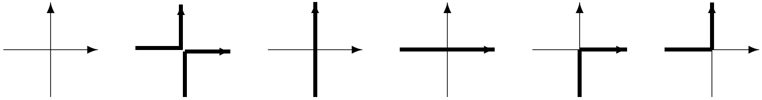
Outline

1. Formulation of q-oscillator lattice
2. Tetrahedron equation
3. Representations of quantum groups

Weights

Let **Z** be a partition function of completely ingomogeneous ice-type model on a rectangular periodical lattice $N \times M$ with the following weights of j^{th} vertex:



$1 \qquad \nu^2 \qquad \lambda \mathbf{k}_j \qquad \mu \mathbf{k}_j \qquad \nu \mathbf{x}_j \qquad \nu \mathbf{y}_j$

Numbers λ, μ $[\nu^2 \equiv -q^{-1}\lambda\mu]$ are the same for all vertices.

$$\mathcal{H} \;:\; \mathbf{xy} = 1 - q^2\mathbf{k}^2 \,, \quad \mathbf{yx} = 1 - \mathbf{k}^2 \,, \quad \mathbf{xk} = q\mathbf{kx} \,, \quad \mathbf{ky} = q\mathbf{yk} \,.$$

Vertex index j stands for a component of tensor power $\mathbf{k}_j, \mathbf{x}_j, \mathbf{y}_j \in \mathcal{H}^{\otimes NM}$.

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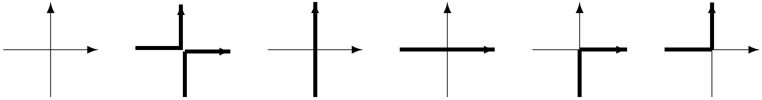

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Main result

Partition function is a polynomial of λ, μ with operator-valued coefficients

$$\mathbf{Z}(\lambda,\mu) = \sum_{n,m=0}^{M,N} \lambda^{Mn} \mu^{Nm} \mathbf{z}_{n,m} \;, \quad \mathbf{z}_{n,m} \in \mathcal{H}^{\otimes NM}$$

Theorem

$$\mathbf{Z}(\lambda,\mu)\mathbf{Z}(\lambda',\mu') = \mathbf{Z}(\lambda',\mu')\mathbf{Z}(\lambda,\mu)$$

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Matrix L

$$L_{\alpha,\beta}(\mathcal{H};\lambda,\mu) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda\mathbf{k} & \nu\mathbf{y} & 0 \\ 0 & \nu\mathbf{x} & \mu\mathbf{k} & 0 \\ 0 & 0 & 0 & \nu^2 \end{pmatrix}, \quad \nu^2 = -q^{-1}\lambda\mu$$

$L_{\alpha,\beta}$ acts in $V_\alpha \otimes V_\beta \otimes F$, $V = \mathbb{C}^2$ and F is e.g. the Fock space $\mathbf{k} = q^{\mathbf{n}}$.

$$L_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \overset{\curvearrowleft}{\prod_n} \overset{\curvearrowleft}{\prod_m} L_{\alpha_n,\beta_m}(\mathcal{H}_{n,m};\lambda,\mu)\,, \quad \mathbf{Z} \;=\; \underset{V_{\boldsymbol{\alpha}},V_{\boldsymbol{\beta}}}{\mathrm{Trace}}\,(L_{\boldsymbol{\alpha},\boldsymbol{\beta}})$$

$$\overset{\curvearrowleft}{\prod_n} f_n = f_N f_{N-1}\ldots f_1,\; V_{\boldsymbol{\alpha}} = \bigotimes_{n=1}^{N} V_{\alpha_n} \text{ etc.}$$

$$M_{\alpha,\alpha'}\left(\mathcal{H}_0;\frac{\mu}{\mu'}\right)M_{\beta,\beta'}\left(\mathcal{H}_0;\frac{\lambda'}{\lambda}\right)L_{\alpha,\beta}(\mathcal{H};\lambda,\mu)L_{\alpha',\beta'}(\mathcal{H};\lambda',\mu')=$$

$$L_{\alpha',\beta'}(\mathcal{H};\lambda',\mu')L_{\alpha,\alpha'}(\mathcal{H};\lambda,\mu)M_{\beta,\beta'}\left(\mathcal{H}_0;\frac{\lambda'}{\lambda}\right)M_{\alpha,\alpha'}\left(\mathcal{H}_0;\frac{\mu}{\mu'}\right)$$

where

$$M(\mathcal{H}_0;\xi)=\begin{pmatrix}\xi^{\mathbf{n}_0} & 0 & 0 & 0\\ 0 & \lambda_0(-q\xi)^{\mathbf{n}_0} & \nu_0\xi^{-1/2+\mathbf{n}_0}\mathbf{y}_0 & 0\\ 0 & \nu_0\xi^{1/2+\mathbf{n}_0}\mathbf{x}_0 & \mu_0(-q\xi)^{\mathbf{n}_0} & 0\\ 0 & 0 & 0 & \nu_0^2\xi^{\mathbf{n}_0}\end{pmatrix}$$

$$\mathbf{k}_0=q^{\mathbf{n}_0},\ \nu_0^2\equiv q^{-1}\lambda_0\mu_0.$$

Lax operators and R matrices

$$\mathcal{L}^{(n)}_{\boldsymbol{\beta}}(\lambda,\mu) \;=\; \operatorname*{Trace}_{V_\alpha} \prod_m^{\curvearrowleft} L_{\alpha_n,\beta_m}(\mathcal{H}_{n,m};\lambda,\mu) \;:\quad V^{\otimes M}\otimes F^{\otimes M}$$

$$V^{\otimes M} = \bigoplus_{m=0}^{M} V_{\pi_m} \;\Rightarrow\; \mathcal{L}_{\boldsymbol{\beta}}(\lambda,\mu) \;=\; \bigoplus_{m=0}^{M} \mu^m \, L_{\pi_m}(\lambda)$$

π_m – m th fundamental representation of $\mathcal{U}_q(\widehat{sl}_M)$, π_0 and π_M – scalars. In quantum space $q^{\mathbf{n}_{n,1}+\mathbf{n}_{n,2}+\cdots\mathbf{n}_{n,M}} = q^{J_n}$ is the center, $F^{\otimes M} = \bigoplus_{J=0}^{\infty} V_{J\pi_1}$.

$$R_{\boldsymbol{\beta},\boldsymbol{\beta}'}\left(\frac{\lambda}{\lambda'}\right) = \operatorname*{Trace}_{F_0} \prod_m^{\curvearrowleft} M_{\beta_m,\beta'_m}\left(\mathcal{H}_0;\frac{\lambda'}{\lambda}\right) = \sum_{m,m'=0}^{M} \lambda_0^m \mu_0^{m'} R_{\pi_m,\pi_{m'}}\left(\frac{\lambda}{\lambda'}\right)$$

Lax operator for $M = 2$

$$\mathcal{L}(\lambda,\mu) = \begin{pmatrix} 1+\lambda^2 q^J & & \\ & \nu^2 L_{1/2}(\lambda) & \\ & & \mu^2(q^J + q^{-2}\lambda^2) \end{pmatrix}$$

where $q^J = q^{\mathbf{n}_1+\mathbf{n}_2}$ is the center and

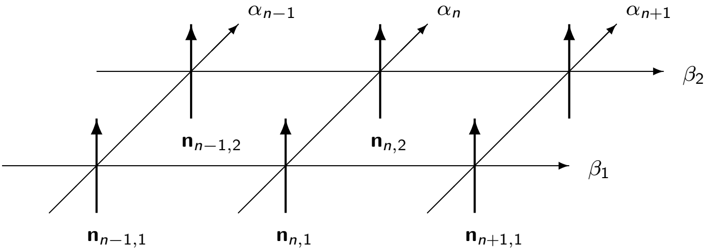
$$L_{1/2}(\lambda) = \begin{pmatrix} \lambda q^{\mathbf{n}_2} - \lambda^{-1} q^{\mathbf{n}_1+1} & \mathbf{x}_1\mathbf{y}_2 \\ \mathbf{y}_1\mathbf{x}_2 & \lambda q^{\mathbf{n}_1} - \lambda^{-1} q^{\mathbf{n}_2+1} \end{pmatrix}$$

In the limit $q \to 1$: $\mathbf{x} \to \mathbf{a}$, $\mathbf{y} \to \mathbf{a}^+$ and

$$J = \mathbf{a}_1^+\mathbf{a}_1 + \mathbf{a}_2^+\mathbf{a}_2\ ,\quad H = \mathbf{a}_1^+\mathbf{a}_1 - \mathbf{a}_2^+\mathbf{a}_2\ ,\quad E = \mathbf{a}_1^+\mathbf{a}_2\ ,\quad F = \mathbf{a}_2^+\mathbf{a}_1$$

Six vertex model

$M = 2$, N – length of $\mathcal{U}_q(\widehat{sl}_2)$ chain.



Integrals of motion: $J_n = \mathbf{n}_{n,1} + \mathbf{n}_{n,2}$, $K_1 = \sum_n \mathbf{n}_{n,1}$, $K_2 = \sum_n \mathbf{n}_{n,2}$.

Six vertex model: $J_n = 1$ for all n (spin $1/2$)

$\Rightarrow K_1$="number of spins up", K_2="number of spins down".

THANK YOU

Universal form of nested Bethe Ansatz

Decomposition of **Z**:

$$\mathbf{Z}(\lambda,\mu) = \sum_{n,m=0}^{M,N} \lambda^{Mn}\mu^{Nm}\mathbf{z}_{n,m}\ , \quad \mathbf{z}_{n,m}\in\mathcal{H}^{\otimes NM}$$

Let $\mathbf{uv} = q^2\mathbf{vu}$ – additional auxiliary Weyl algebra and let

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \sum_{n,m}(-q)^{-nm}\mathbf{u}^n\mathbf{v}^m z_{n,m}$$

Universal NBA equation:

$$\langle Q|\mathbf{J}(\mathbf{u},\mathbf{v}) = 0$$

E.g. $Q(u) = \langle Q|u\rangle$ etc.

Baxter's equation for $\mathcal{U}_q(\widehat{sl}_2)$

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \sum_{n,m} (-q)^{-nm} \mathbf{u}^n \mathbf{v}^m z_{n,m} \; : \quad \langle Q | \mathbf{J}(\mathbf{u},\mathbf{v}) = 0$$

For $M = 2$ $J(\mathbf{u},\mathbf{v}) = \phi(q^2\mathbf{u}) - t(\mathbf{u})\mathbf{v} + \phi'(q^{-2}\mathbf{u})\mathbf{v}^2$

$$\langle Q|u\rangle = Q(u), \quad \langle Q|\mathbf{u}|u\rangle = uQ(u), \quad \langle Q|\mathbf{v}|u\rangle = vQ(q^2u)$$

$$\langle Q|J(\mathbf{u},\mathbf{v})|q^{-2}u\rangle \equiv \phi(u)Q(q^{-2}u) - t(u)Q(u)v + \phi'(u)Q(q^2u)v^2 = 0$$

Fix v by condition $Q(0) = 1$, then $Q(u)$ is a polynomial.