Asymptotic spectrum of a lattice spin system

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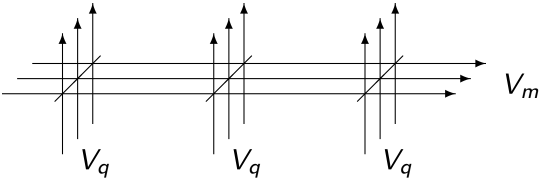
$\mathcal{U}_{q=-1}(\widehat{gl}_M)$ chain of the length N.



- V_q is the minimal cyclic representation of $\mathcal{U}_q(\widehat{gl}_M)$ with $\mathbb{Z}_M$ symmetry (only one parameter or representation survives). $Dim(V_q) = 2^M$
- V_m is the m^{th} fundamental representation of $\mathcal{U}_q(\widehat{gl}_M)$
- N is the length of the chain
- Transfer matrix $t_m(y)$, $m = 0, 1, \ldots, M$.

$$t_m(y) = \sum_{n=0}^{N} y^n t_{m,n}$$

Rank-size $N \leftrightarrow M$ duality.

- The complimentary summation $\sum_{m=0}^{M} x^m t_{m,n} = t'_n(x)$ gives the transfer matrices for the homogeneous $\mathcal{U}_q(\widehat{gl}_N)$ chain of the length M
- Quantum transfer matrix of the generalized chiral Potts model is the layer-to-layer transfer matrix of $3d$ Zamolodchikov model
- Operators $t_{m,n} = t^{\dagger}_{m,n}$ have a combinatorial representation in the terms of interacting spins (Pauli matrices) situated in the vertices of $N \times M$ square lattice

$$\{t_{m,n}\}\,, \quad 0 \le m \le M\,,\; 0 \le n \le N\,,\; N, M \to \infty$$

Spectral equation

Bi-spectral transfer matrices

$$t_{\alpha,\beta}(x,y) = \sum x^{2m+\alpha} y^{2n+\beta} t_{2m+\alpha,2n+\beta}\ , \quad \alpha,\beta = 0,1$$

obey the spectral equation

$$t_{0,0}(x,y)^2 - t_{1,0}(x,y)^2 - t_{0,1}(x,y)^2 - t_{1,1}(x,y)^2 \ = \ F(x^2,y^2)\ .$$

$$F(x^2,y^2) \ = \ \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (1 - \lambda \mathrm{e}^{2\pi i \frac{n}{N}} - \mu \mathrm{e}^{2\pi i \frac{m}{M}} - \kappa^2 \lambda\mu \mathrm{e}^{2\pi i (\frac{n}{N} + \frac{m}{M})})$$

Here $\lambda^N = x^2$, $\mu^M = y^2$, and κ is the single parameter of $\mathbb{Z}_M, \mathbb{Z}_N$ - homogeneous cyclic representation.

x^2, y^2-decomposition $\mapsto$ Abelian algebra for $t_{m,n}$.

Comment on the spectral equation

Let $M = 2$ (Ising model). Baxter's equation

$$t(y)Q(y) = \phi(y)Q(qy) + \phi'(y)Q(q^{-1}y)\ , \quad q^2 = 1$$

is equivalent to

$$\det \begin{vmatrix} t(y) & \phi(y) + \phi'(y) \\ \phi(-y) + \phi'(-y) & t(-y) \end{vmatrix} = 0\ .$$

Then, introducing the second spectral parameter x, we come to the spectral equation in our form:

$$\det \begin{vmatrix} t(y) & x\phi(y) + x^{-1}\phi'(y) \\ x\phi(-y) + x^{-1}\phi'(-y) & t(-y) \end{vmatrix} =$$

$$(1 - x^2)\phi(y)\phi(-y) + (1 - x^{-2})\phi'(y)\phi'(-y) \equiv x^{-2}F(x^2, y^2)\ .$$

- We analyze the eigenvalues of $\{t_{m,n}\}$ when $N, M \to \infty$, $\zeta = N/M$ is nonsingular.
- Notations: let

$$\Pi = \{(m,n)\} \;:\; 0 \leq m \leq M,\; 0 \leq n \leq N\;,$$

$$c = \cot(a/2) = \sqrt{\frac{1+\kappa^2}{3-\kappa^2}}\;,$$

$$\Omega(p,q) \;=\; \frac{\pi}{2}\left(\zeta\frac{1+c^2}{2c}p^2 + \frac{1-c^2}{c}pq + \zeta^{-1}\frac{1+c^2}{2c}q^2\right)\;.$$

- The biggest $t_{m,n}$ always belong to the "middle" of Π:

$$P_0 = [M(1-a/\pi)]\;,\quad Q_0 = [N(1-a/\pi)]\;.$$

- Elements $t_{m,n}$ far from the boundary of Π always have the form

$$t_{\underbrace{P_0+p}_{m},\underbrace{Q_0+q}_{n}} = f_0 \; \mathrm{e}^{NMg_0} \mathrm{e}^{-\Omega(p,q)/2} \; c_{p,q} \; \cdot \; (1+O(1/NM))$$

- $c_{p,q}$ for the excited states may be found from the asymptotical spectral equation

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\Omega(m,n)} c_{p-m,q-n} c_{p+m,q+n} \; = \; 1 \quad \forall \; p,q \in \mathbb{Z} \, .$$

- For the ground state

$$c_{p,q} = c_0 \sqrt{NM}$$

THANK YOU

Bibliography I

- *J. Phys. A: Math. Gen.* **32** (1999) 5693–5714
- *Theoretical and Mathematical Physics* **124** (2000) 391–409
- *Journal of Mathematical Sciences* **115**(1) (2003) 2049-2057
- *Theoretical and Mathematical Physics* **138** (2004) 310-321
- *Theoretical and Mathematical Physics* **138** (2004) 226-237
- *Particles and Nuclei* **35** (2004) 1051-1111.
- *Preprint* arXiv:nlin.SI/0506022