Spectrum of a lattice spin system

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Outline

Zamolodchikov model.

- Integrability of Zamolodchikov model is based on the existence of commutative set of transfer matrices

$$\left[T(\theta_1,\theta_2,\theta_3),\,T(\theta_1,\theta_2',\theta_3')\right] \;=\; 0$$

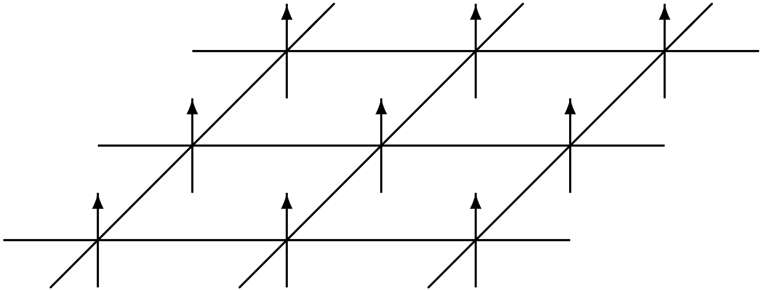
- T is the layer-to-layer transfer matrix, if the layer of the system has the size $N \times M$ then T is $2^{NM} \times 2^{NM}$ matrix.
- Commutativity of transfer matrices means that there exists a decomposition

$$T(\theta_1,\theta_2,\theta_3) \;=\; \sum_{m,n} t_{m,n}(\theta_1)\chi_{m,n}(\theta_1,\theta_2,\theta_3)$$

with numerical coefficients $\chi_{m,n}$ and commutative moduli $t_{m,n}(\theta_1)$.

Graphical view of layer-to-layer matrix

lyrical digression



Disadvantage of Zamolodchikov model

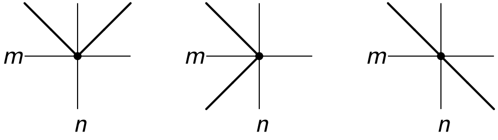
- Boltzmann weights are not positively defined (moreover, they are complex).
- There was no method for the investigation of the whole spectrum of T.
- Nested Bethe ansatz approach for $N \to \infty$ and $M < \infty$ is not 3D-invariant since $M/N \to 0$.

Issue is the analysis of the structure of $t_{m,n}(\theta_1)$ – the spin lattice system.

Spin Lattice system.

S. Sergeev, JPA **32** (1999) 5693–5714, TMP **124** (2000) 391–409, JMS **115** (2003) 2049-2057, TMP **138** (2004) 310-321, TMP **138** (2004) 226-237, PN **35** (2004) 1051-1111.

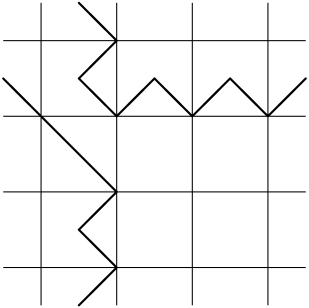
Combinatorial formulation: Consider a system of paths on $N \times M$ lattice with the rules of bypassing a vertex:



$$\gamma_{n,m} = \sigma^x_{n,m} \qquad \gamma_{n,m} = \sigma^y_{n,m} \qquad \gamma_{n,m} = \kappa\sigma^z_{n,m}$$

The "weights" $\sigma^x_{n,m}$, $\sigma^y_{n,m}$ and $\sigma^z_{n,m}$ are the local Pauli matrices.

Example of a path
lyrical digression



Periodical boundary conditions are implied, any path is characterized by the winding numbers.

Commutative operators

- Any path $\mathcal{P}$ has a homotopy class $c(\mathcal{P}) = m\mathcal{A} + n\mathcal{B}$
- For fixed winding numbers n and m let

$$J_{m,n}(\kappa) = \sum_{\mathcal{P}\,:\,c(\mathcal{P})=m\mathcal{A}+n\mathcal{B}} \prod_{\mathcal{P}} \gamma_V$$

- self-adjoint operators $J_{m,n}$ obey the following exchange relations:

$$J_{m,n}J_{m',n'} = (-)^{nm'+n'm} J_{m',n'}J_{m,n} .$$

- $J_{m,n}(\kappa) = \mathrm{i}^{nm} (\sigma^x)^m (\sigma^y)^n t_{m,n}(\kappa) , \quad \left[t_{m,n}, t_{m',n'}\right] = 0 .$

Relation to T
lyrical digression

Both $J_{m,n}$ and $t_{m,n}$ are the "integrals of motion" of the Zamolodchikov model:

$$[\, J_{m,n}(\kappa)\,,\ T(\theta_1,\theta_2,\theta_3)\,] \;=\; [\, t_{m,n}(\kappa)\,,\ T(\theta_1,\theta_2,\theta_3)\,] \;=\; 0$$

if

$$\kappa \;=\; \tan\frac{\theta_1}{2}$$

We prefer the title moduli for $t_{m,n}$ since in the classical limit they become the moduli of the spectral curve.

Finite size spectral equation

$$J(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{N} (-)^{n+m+nm} x^m y^n J_{m,n}$$

$$= t_{0,0}(x,y) - \sigma^x t_{1,0}(x,y) - \sigma^y t_{0,1}(x,y) - \sigma^z t_{1,1}(x,y)$$

Generating function obeys

$$t_{0,0}(x,y)^2 - t_{1,0}(x,y)^2 - t_{0,1}(x,y)^2 - t_{1,1}(x,y)^2 = F(x^2,y^2)\,,$$

where

$$F(X,Y) = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (1 - \lambda \mathrm{e}^{2\pi i n/N} - \mu \mathrm{e}^{2\pi i m/M} - \kappa^2 \lambda\mu \mathrm{e}^{2\pi i (n/N+m/M)})$$

$\lambda^N = X$, $\mu^M = Y$.

The leading term

$$|t_{\alpha,\beta}(x,y)|^2 \sim |F(x^2,y^2)| \approx \mathrm{e}^{NM\mathfrak{g}(\lambda,\mu;\kappa^2)}, \quad x^2=\lambda^N, \quad y^2=\mu^M$$

$$\mathfrak{g}(\lambda,\mu;\kappa^2) = \frac{1}{(2\pi)^2}\int\int_0^{2\pi} d\phi d\phi' \log\left|1-\lambda\mathrm{e}^{\mathrm{i}\phi}-\mu\mathrm{e}^{\mathrm{i}\phi'}-\kappa^2\lambda\mu\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right| ,$$

$$|\lambda| = \frac{\sin r_2}{\sin r_1}, \quad |\mu| = \frac{\sin r_3}{\sin r_1}, \quad \kappa^2 = \frac{\sin r_0 \sin r_1}{\sin r_2 \sin r_3},$$

$$\mathfrak{g}(\lambda,\mu;\kappa^2) = -\log 2\sin r_1 + \sum_{j=0}^{3}\left(\frac{r_j}{\pi}\log 2\sin r_j + \Phi(r_j)\right) ,$$

$$\Phi(x) = \sum_{m=1}^{\infty}\frac{\sin(2mx)}{2\pi m^2} .$$

Abelian Algebra of $t_{m,n}$

The leading term is too rough approximation: it is the same for all eigenvalues.
$F(X, Y)$ is a polynomial,

$$F(X, Y) = \sum_{P=0}^{M} \sum_{Q=0}^{N} X^P Y^Q F_{P,Q} ,$$

the spectral equations are just a set of bilinear relations:

$$\sum_{m,n} (-)^{m+n+mn+mQ+nP} t_{m,n} t_{2P-m,2Q-n} = F_{P,Q} .$$

These equations are just the Abelian algebra for $t_{m,n}$.

- Finite N, M:
  - Excluding the variables $t_{m,n}$, one comes finally to 2^{NM}-power polynomial for a single variable
  - this means that the Abelian algebra is completely equivalent to the original spin-lattice problem
  - At $N = M = 6$ it is possible to solve all equations, although $2^{36} = 68719476736$, but higher N, M are hopeless
- To solve these equations for finite M with $N \to \infty$
  - loss of generality, singular aspect ratio
- The final idea is to cut the $\mathrm{e}^{NM\mathfrak{g}}$ term and derive limiting $N, M \to \infty$ equations for "fine structure" of $t_{m,n}$.

Define

$$c = \cot\frac{a}{2} = \sqrt{\frac{1+\kappa^2}{3-\kappa^2}}\ .$$

Let

$$\Omega(p,q) = \frac{\pi}{2}\left(\zeta\frac{1+c^2}{2c}p^2 + \frac{1-c^2}{c}pq + \zeta^{-1}\frac{1+c^2}{2c}q^2\right)\ .$$

where $\zeta = N/M$ – the aspect ratio. Let P_0, Q_0 be the even numbers such that

$$M\cdot\left(1-\frac{a}{\pi}\right) = P_0 - u_1\ ,\quad N\cdot\left(1-\frac{a}{\pi}\right) = Q_0 - u_2$$

(P_0 and Q_0 are the “middle” of the domain of $F_{P,Q}$) The fractional parts obey $-1 < u_1, u_2 \leq 1$.

Coefficients $F_{P,Q}$ have the following asymptotics:

$$F_{P_0+p,Q_0+q} = \mathrm{e}^{NMg(\kappa^2)} f_0 \cdot (-)^{p+q+pq} \mathrm{e}^{-\Omega(p+u_1,q+u_2)} \left(1 + O(\frac{1}{NM})\right)$$

where

$$g(\kappa^2) \equiv \mathfrak{g}(1,1;\kappa^2) = \left(1 - \frac{3a}{2\pi}\right) \log \kappa^2 + 3\Phi\left(\frac{a}{2}\right) - \Phi\left(\frac{3a}{2}\right) .$$

Asymptotic equation for $t_{m,n}$

Define next

$$t_{P_0+m,Q_0+n} \;=\; \sqrt{f_0}\, \mathrm{e}^{\frac{NM}{2} g(\kappa^2)}\, \tau_{m,n}\;,$$

where the leading m, n-independent term is taken into account, and moreover

$$\tau_{m,n} \;=\; c_{m,n}\, \mathrm{e}^{-\frac{1}{2}\Omega(m+u_1,n+u_2)}$$

Then the equation for $c_{m,n}$ is

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\Omega(n,m)} c_{p-m,q-n} c_{p+m,q+n} \;=\; 1 \;\;\forall\;\; p, q \;\in\; \mathbb{Z}\;.$$

This equation contains no N and M, this is THE EQUATION for investigation.

Analysis

The renormalized equation

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\beta\Omega(n,m)} c_{p-m,q-n} c_{p+m,q+n} \;=\; 1 \quad \forall\, p,q \,\in\, \mathbb{Z}\,,$$

with $\beta > 1$ may be solved in the terms of perturbation theory in $Q = \mathrm{e}^{-\beta\Omega(1,0)}$ and $\widetilde{Q} = \mathrm{e}^{-\beta\Omega(0,1)}$: in zero order

$$c_{p,q}^2 = 1 + o(1) \quad \Rightarrow \quad c_{p,q} \;=\; \varepsilon_{p,q}\,(1 + o(1))\,,$$

where $\varepsilon_{p,q}$ are the signs, in the few first orders

$$c_{p,q} \;=\; \varepsilon_{p,q}\,\Big(1 + (\varepsilon_{p+1,q}\varepsilon_{p-1,q} + \varepsilon_{p,q-1}\varepsilon_{p,q+1}) Q\widetilde{Q} + (\ldots) Q^2\widetilde{Q}^2 + \ldots$$

$$+ \varepsilon_{p-1,q+1}\varepsilon_{p+1,q-1}\widetilde{Q}^4 + \varepsilon_{p-1,q-1}\varepsilon_{p+1,q+1} Q^4 + \ldots\Big)\,.$$

Analysis
continued

Conjecture

The seria for $c_{m,n}$ are convergent for $\beta > 1$

If $\beta \to 1$ and all $\varepsilon_{p,q} = (+)$,

$$c_{p,q} = c_0 = \left(\sum_{m,n} (-)^{m+n+nm} \mathrm{e}^{-\beta\Omega(m,n)}\right)^{-1/2} \approx \sqrt{\frac{1}{(\beta-1)\chi}}$$

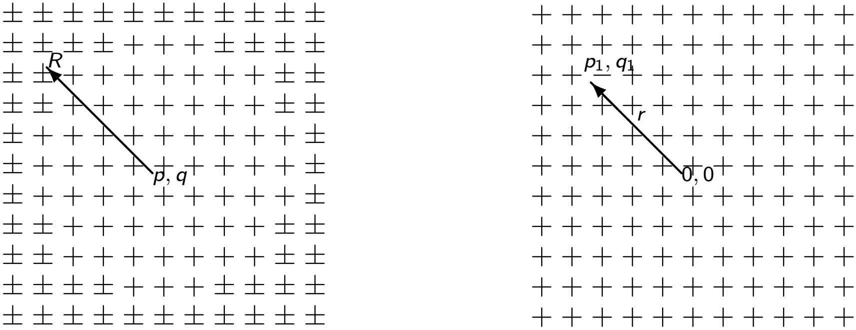
Finite size corrections (more detailed estimation of $F_{P,Q}$) give

$$\beta = 1 + \frac{\gamma}{NM} \Rightarrow c_0 \sim \sqrt{NM} .$$

Conjecture

The seria for $c_{m,n}$ with non-constant $\varepsilon_{p,q}$ are convergent for $\beta = 1$

Cut-off



$c_{p,q} \sim R$ $\qquad$ $c_{0,0} \sim r$

Candidates for a Hamiltonian
qualitative results

$$\tau_{p,q} = c_{p,q}\, \mathrm{e}^{-\frac{1}{2}\Omega(p+u_1,q+u_2)}$$

The first candidate for a Hamiltonian is

$$H = -\sum_{p,q} \tau_{p,q}^2 \quad \rightarrow \quad \text{finite density} \quad h = -\frac{H}{V} \,.$$

The sea of $\varepsilon_{p,q} = (+)$ with the impurities

$$\varepsilon_{p_1,q_1} = \varepsilon_{p_2,q_2} = \cdots = \varepsilon_{p_n,q_n} = (-)$$

is the *n*-particle state of the model.
For one particle, one has gap-less excitations for $\frac{p_1}{R}, \frac{q_1}{R} \sim 1$.

Candidates for a Hamiltonian

alternative variants

Alternative variant:

$$H' = + \sum_{p,q} \tau_{p,q}^2$$

Then $[\varepsilon_{p,q} = \mathrm{stocastic}]$ may be considered as the ground state,
The islands of the definitely positive signs $\varepsilon_{p,q}$ are the bound states.

Alternative variant for the dispersion relation:

$$P_{\alpha,\beta} = \left| \sum_{p,q} \tau_{2p+\alpha,2q+\beta} \right| , \quad P_0^2 = \vec{P}^2$$

parameter for the dispersion relation is a point on the sphere $\vec{n} = \dfrac{\vec{P}}{P_0}$.

The main results:

- Ground state distribution

$$\tau_{p,q} \;=\; c_0\, \mathrm{e}^{-\frac{1}{2}\Omega(p+u_1,q+u_2)}\;, \quad c_0 \;\sim\; \sqrt{V}\;.$$

- Equation for $c_{p,q}$ for excited states

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\Omega(n,m)} c_{p-m,q-n} c_{p+m,q+n} \;=\; 1 \quad \forall\; p,q \;\in\; \mathbb{Z}\;,$$

  Its solution is defined by the distribution of sings of $c_{p,q}$

- Mathematical problem for the further investigation – how to solve equation for $c_{p,q}$.

THANK YOU

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R-matrix

Notations: θ_i–dihedral angles, a_i–sides, β_i–linear excesses:

$$\cos a_i \;=\; \frac{\cos\theta_i+\cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k}\;,\quad \cos\theta_i \;=\; \frac{\cos a_i-\cos a_j\cos a_k}{\sin a_j\sin a_k}\;,$$

$$\beta_0=\pi-\frac{a_1+a_2+a_3}{2}\;,\quad \beta_i=\pi-\beta_0-a_i\;,\quad \sqrt{\tan\frac{\beta_i}{2}}=t_i\;.$$

$R=$

$$\begin{pmatrix}
1 & 0 & 0 & -t_2t_3 & 0 & it_1t_3 & t_1t_2 & 0\\
0 & t_0t_1t_2t_3 & t_0t_1 & 0 & it_0t_2 & 0 & 0 & t_0t_3\\
0 & t_0t_1 & t_0t_1t_2t_3 & 0 & t_0t_3 & 0 & 0 & it_0t_2\\
-t_2t_3 & 0 & 0 & 1 & 0 & t_1t_2 & it_1t_3 & 0\\
0 & -it_0t_2 & t_0t_3 & 0 & t_0t_1t_2t_3 & 0 & 0 & -t_0t_1\\
-it_1t_3 & 0 & 0 & t_1t_2 & 0 & 1 & t_2t_3 & 0\\
t_1t_2 & 0 & 0 & -it_1t_3 & 0 & t_2t_3 & 1 & 0\\
0 & t_0t_3 & -it_0t_2 & 0 & -t_0t_1 & 0 & 0 & t_0t_1t_2t_3
\end{pmatrix}$$

Derivation of the asymptotic of $F_{P,Q}$

Evidently,

$$F_{P,Q} \;=\; \frac{1}{(2\pi \mathrm{i})^2} \oint\oint \frac{dX}{X}\frac{dY}{Y}\frac{F(X,Y)}{X^P Y^Q}\;.$$

Let

$$\alpha_p \;=\; \frac{P\pi}{M}\;,\quad \alpha_q \;=\; \frac{Q\pi}{N}\;.$$

Then

$$\log\left(\frac{F(X,Y)}{X^P Y^Q}\right) \;\sim\; NM\left(\mathfrak{g}(\lambda,\mu;\kappa^2) - \frac{\alpha_p}{\pi}\log\lambda - \frac{\alpha_q}{\pi}\log\mu\right)$$

has the extremum at

$$r_0 + r_2 \;=\; \alpha_p\;,\quad r_0 + r_3 \;=\; \alpha_q\;.$$

Derivation of the asymptotic of $F_{P,Q}$
continued

Extremal value is

$$g(\alpha_p, \alpha_q; \kappa^2) \;=\; \frac{r_0}{\pi} \log \kappa^2 + \sum_{j=0}^{3} \Phi(r_j)$$

The variation

$$g(\alpha_p, \alpha_q; \kappa^2) \;=\; g_0(\kappa^2) \;-\; \frac{1+c^2}{4\pi c}(\delta\alpha_p^2 + \delta\alpha_q^2) \;-\; \frac{1-c^2}{2\pi c}\delta\alpha_p \delta\alpha_q \;,$$

where

$$\delta\alpha_p \;=\; \frac{\pi}{M}(p+u_1)\;, \quad \delta\alpha_q \;=\; \frac{\pi}{N}(q+u_2)\;.$$

gives

$$F_{P_0+p,Q_0+q} \;=\; (-)^{p+q+pq} \;\cdot\; f_0 \;\cdot\; \mathrm{e}^{NMg_0(\kappa^2)-\Omega(p+u_1,q+u_2)} \;,$$

Explanation of the finite size spectral equation
In the terms of Bethe ansatz

$$t(x)Q(x) = \phi(x)Q(qx) + \phi'(x)Q(q^{-1}x)\,, \quad q^2 = 1\,.$$

$$\det \begin{vmatrix} t(x) & \phi(x) + \phi'(x) \\ \phi(-x) + \phi'(-x) & t(-x) \end{vmatrix} = 0\,.$$

$$\det \begin{vmatrix} t(x) & y\phi(x) + y^{-1}\phi'(x) \\ y\phi(-x) + y^{-1}\phi'(-x) & t(-x) \end{vmatrix} =$$

$$(1 - y^2)\phi(x)\phi(-x) + (1 - y^{-2})\phi'(x)\phi'(-x) \equiv F(x^2, y^2)\,.$$