New Solutions Of The Tetrahedron Equation

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Outline

1. Simplex equations
2. Local Yang-Baxter equation
3. Quantization scheme
4. Main Example
5. Quantization
6. Representation
7. Newly obtained model from 2D approach
8. Summary

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Simplex equations

The Yang-Baxter Equation

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

The tetrahedron Equation,

$$R_{1,2,3}R_{1,4,5}R_{2,4,6}R_{3,5,6} = R_{3,5,6}R_{2,4,6}R_{1,4,5}R_{1,2,3}$$

The 4-simplex equation

$$R_{0,1,2,3}R_{0,4,5,6}R_{1,4,7,8}R_{2,5,7,9}R_{3,6,8,9} = \mathrm{reverse}$$

Etc.

Simplex equations

The Yang-Baxter Equation

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

The tetrahedron Equation,

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The 4-simplex equation

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Etc.

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The 4-simplex equation

$$R_{0,1,2,3}R_{0,4,5,6}R_{1,4,7,8}R_{2,5,7,9}R_{3,6,8,9} = \text{reverse}$$

Etc.

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Etc.

Auxiliary relations

$$R_{1,2}L_{1;\alpha}L_{2;\alpha} = L_{2;\alpha}L_{1;\alpha}R_{1,2}$$

YBE is two ways to intertwine

$$Y \cdot L_{1;\alpha}L_{2;\alpha}L_{3;\alpha} = L_{3;\alpha}L_{2;\alpha}L_{1;\alpha} \cdot Y$$

$$R_{1,2,3}L_{1;\alpha,\beta}L_{2;\alpha,\gamma}L_{3;\beta,\gamma} = L_{3;\beta,\gamma}L_{2;\alpha,\gamma}L_{1;\alpha,\beta}R_{1,2,3}$$

TE is two ways to intertwine

$$T \cdot L_{1;\alpha,\beta}L_{2;\alpha,\gamma}L_{3;\beta,\gamma}L_{4;\alpha,\delta}L_{5;\beta,\delta}L_{6;\gamma,\delta} =$$

$$L_{6;\gamma,\delta}L_{5;\beta,\delta}L_{4;\alpha,\delta}L_{3;\beta,\gamma}L_{2;\alpha,\gamma}L_{1;\alpha,\beta} \cdot T$$

Auxiliary relations

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The Local Yang-Baxter equation

Let $L_{\alpha,\beta}(v) \in \mathrm{End}\,(V_\alpha \otimes V_\beta)$, $v \in \mathbb{C}$ or $\mathbb{C}^2 \ldots$
Let $v'_j = f_j(v_1, v_2, v_3)$, $j = 1, 2, 3$, be the solution of

$$L_{\alpha,\beta}(v_1)L_{\alpha,\gamma}(v_2)L_{\beta,\gamma}(v_3) = L_{\beta,\gamma}(v'_3)L_{\alpha,\beta}(v'_2)L_{\alpha,\beta}(v'_1)$$

Define the functional operator $\mathfrak{R}$: $\forall\ \phi = \phi(v_1, v_2, v_3)$ let

$$(\mathfrak{R}_{1,2,3} \circ \phi)\,(v_1, v_2, v_3) \ = \ \phi(v'_1, v'_2, v'_3)$$

Local YBE takes the form

$$L_{\alpha,\beta}(v_1)\ L_{\alpha,\gamma}(v_2)L_{\beta,\gamma}(v_3) \ = \ \mathfrak{R}_{1,2,3} \circ L_{\beta,\gamma}(v_3)L_{\alpha,\beta}(v_2)L_{\alpha,\beta}(v_1)$$

Then $\mathfrak{R}$ obeys the Functional TE

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Quantization Scheme

Suppose

- Vertex variable v is a list of scalar variables $v = (x, y, \ldots)$.
- Vertex variables v_j have a Poisson structure conserved by the mapping $\mathfrak{R}$
- Poisson algebra may be quantized: v_j becomes an algebra of observables $\mathfrak{v}_j$
- Mapping $\mathfrak{R}$ may be re-calculated as an automorphism of the algebra of observables.

Then for any irreducible representation of the algebra of observables the automorphism must be the internal one:

$$\mathfrak{R}_{1,2,3} \circ \phi \;=\; R_{1,2,3} \cdot \phi \cdot R_{1,2,3}^{-1}$$

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New Example

$L(v,s) \in \mathrm{End}(\mathbb{C}^2 \times \mathbb{C}^2)$,
vertex dynamical variable $v = (x, y)$,
vertex parameters $s = (\lambda, \mu)$

$$L(v;s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda k & y & 0 \\ 0 & -\lambda\mu x & \mu k & 0 \\ 0 & 0 & 0 & -\lambda\mu \end{pmatrix},$$

where $k^2 = 1 - xy$.

$$L_{\alpha,\beta}(v_1; s_1) L_{\alpha,\gamma}(v_2; s_2) L_{\beta,\gamma}(v_3; s_3) =$$

$$L_{\beta,\gamma}(v_3'; s_3) L_{\alpha,\gamma}(v_2'; s_2) L_{\alpha,\beta}(v_1'; s_1)$$

Poisson structure:

$$\begin{cases} \{x_j, y_j\} = 1 - x_j y_j \\ \text{any other bracket is zero.} \end{cases}$$

Quantization

Poisson algebra $\{x, y\} = 1 - xy$ is $q \mapsto 1$ limit of q-oscillator algebra for $\mathbf{x}, \mathbf{y}$:

$$\mathbf{xy} = 1 - q^2\mathbf{k}^2, \quad \mathbf{yx} = 1 - \mathbf{k}^2 \ ,$$

$$\mathbf{xk} = q\mathbf{kx}, \quad \mathbf{ky} = q\mathbf{yk}.$$

Quantum L is given by

$$L(\mathfrak{v}; s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda\mathbf{k} & \mathbf{y} & 0 \\ 0 & -q^{-1}\lambda\mu\mathbf{x} & \mu\mathbf{k} & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda\mu \end{pmatrix}$$

where $\mathfrak{v} = (\mathbf{x}, \mathbf{y}, \mathbf{k})$ and $s = (\lambda, \mu)$.

Explicit form of the quantum mapping

The quantum mapping $\mathfrak{v}^{\otimes 3} \longrightarrow \mathfrak{v}^{\otimes 3}$:

$$\mathbf{k}_2'\mathbf{y}_1' = \tfrac{\lambda_3}{\lambda_2}\left(\mathbf{k}_3\mathbf{y}_1 - \tfrac{\lambda_1\mu_3}{q}\mathbf{k}_1\mathbf{y}_2\mathbf{x}_3\right), \quad \mathbf{k}_2'\mathbf{x}_1' = \tfrac{\lambda_2}{\lambda_3}\left(\mathbf{k}_3\mathbf{x}_1 - \tfrac{q}{\lambda_1\mu_3}\mathbf{k}_1\mathbf{x}_2\mathbf{y}_3\right),$$

$$\mathbf{y}_2' = \mathbf{y}_1\mathbf{y}_3 + \lambda_1\mu_3\mathbf{k}_1\mathbf{k}_3\mathbf{y}_2, \qquad \mathbf{x}_2' = \mathbf{x}_1\mathbf{x}_3 + \frac{q^2}{\lambda_1\mu_3}\mathbf{k}_1\mathbf{k}_3\mathbf{x}_2,$$

$$\mathbf{k}_2'\mathbf{y}_3' = \tfrac{\mu_1}{\mu_2}\left(\mathbf{k}_1\mathbf{y}_3 - \tfrac{\lambda_1\mu_3}{q}\mathbf{k}_3\mathbf{x}_1\mathbf{y}_2\right), \quad \mathbf{k}_2'\mathbf{x}_3' = \tfrac{\mu_2}{\mu_1}\left(\mathbf{k}_1\mathbf{x}_3 - \tfrac{q}{\lambda_1\mu_3}\mathbf{k}_3\mathbf{y}_1\mathbf{x}_2\right).$$

with $\mathbf{k}_j'^2 = \dfrac{\mathbf{x}_j'\mathbf{y}_j' - \mathbf{y}_j'\mathbf{x}_j'}{1-q^2}$, $\mathbf{k}_1'\mathbf{k}_2' = \mathbf{k}_1\mathbf{k}_2$, $\mathbf{k}_2'\mathbf{k}_3' = \mathbf{k}_2\mathbf{k}_3$

Fock space representation

In the basis $\mathbf{x}|0\rangle = 0$, $\mathbf{y}|n\rangle = (1-q^{2n+2})|n+1\rangle$, $\mathbf{k}|n\rangle = q^n|n\rangle$, $\langle m|n\rangle = \delta_{m,n}$,

$$\langle m_1, m_2, m_3|R|n_1, n_2, n_3\rangle = \left(\frac{\lambda_3}{\lambda_2}\right)^{n_1}\left(-\frac{\lambda_1\mu_3}{q}\right)^{m_2}\left(\frac{\mu_1}{\mu_2}\right)^{n_3} r^{n_1,n_2,n_3}_{m_1,m_2,m_3}$$

with

$$r^{n_1,n_2,n_3}_{m_1,m_2,m_3} = \delta_{m_1+m_2,n_1+n_2}\delta_{m_2+m_3,n_2+n_3}\frac{q^{(n_1-m_2)(n_3-m_2)}}{(q^2;q^2)_{n_2}}P_{n_2}(q^{2m_1}, q^{2m_2}, q^{2m_3})$$

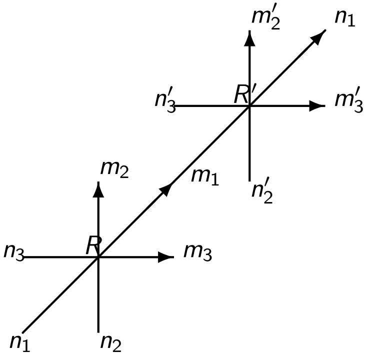
Polynomial P defined recursively,

$$P_n(x,y,z) = (1-x)(1-z)P_{n-1}\left(\frac{x}{q^2}, y, \frac{z}{q^2}\right) - q^{2-2n}xz(1-y)P_{n-1}\left(x, \frac{y}{q^2}, z\right)$$

with $P_0(x,y,z) = 1$.

Back to 2D – general concept

Any solution of TE produces a series of solutions of YBE (n layers).



$$\sum_{n_1, m_1} R_{m_1, m_2, m_3}^{n_1, n_2, n_3} R'^{m_1, n_2', n_3'}_{n_1, m_2', m_3'} = \mathcal{R}_{(m_2, m_2'), (m_3, m_3')}^{(n_2, n_2'), (n_3, n_3')}$$

Back to 2D – two layers

Delta-functions in $\mathcal{R}_{(m_2,m_2'),(m_3,m_3')}^{(n_2,n_2'),(n_3,n_3')}$ are

$$\underbrace{n_2+n_2'=m_2+m_2'}_{j_2} \quad \text{and} \quad \underbrace{n_3+n_3'=m_3+m_3'}_{j_3}$$

$$\mathcal{R}_{(m_2,j_2-m_2),(m_3,j_3-m_3)}^{(n_2,j_2-n_2),(n_3,j_3-n_3)} = \alpha^{j_2}\beta^{j_3}\left(R^{(j_2,j_3)}\left(q^{j_3-j_2}\frac{\lambda_3\lambda_3'}{\lambda_2\lambda_2'}\right)\right)_{m_2,m_3}^{n_2,n_3}$$

$$0\le n_2,m_2\le j_2\ ,\quad 0\le n_3,m_3\le j_3$$

$R^{(j_2,j_3)}$: $\frac{j_2}{2}\times\frac{j_3}{2}$ representation of $\mathcal{U}_q(\widehat{sl}_2)$. E.g. $j_2=1$ and $j_3=\ell$:

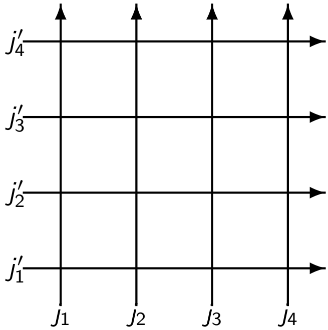
$$R_{0,m}^{0,m}(z)=\frac{q^{m-\ell+1}(z-q^{2\ell-2m})}{(1-z)(1-q^2z)},\quad R_{1,m}^{1,m}(z)=\frac{q^{1-m}(z-q^{2m})}{(1-z)(1-q^2z)},$$

$$R_{0,m+1}^{1,m}(z)=\frac{q^{1-\ell}(1-q^{2m+2})z}{(1-z)(1-q^2z)},\quad R_{1,m}^{0,m+1}(z)=\frac{(1-q^{2\ell-2m})}{(1-z)(1-q^2z)}.$$

Back to 2D – interpretation of 3D model

Partition function for $R_{1,2,3}$ in two layers is

$$Z = \sum_{\text{all } j_n, j'_m \geq 0} Z^{(6V)}_{\{j\},\{j'\}} f_{\{j\},\{j'\}}$$



Summary

- "Partition function" of quantum $L_{\alpha,\beta}$ produces the complete set of commutative polynomials of q-oscillator generators.
- $n > 2$ layers of bosonic $R_{1,2,3}$ give $\mathcal{U}_q(\widehat{sl}_n)$ R-matrices for all symmetric tensor representations.
- Cyclic representation of the q-oscillator algebra with q^2–odd root of unity: dihedral angles parameterization for simple TE, algebraic geometry parameterization for modified TE.
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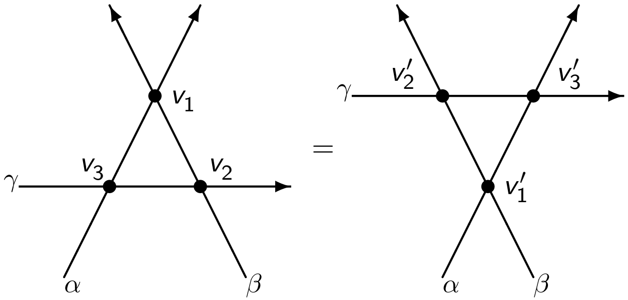
THANK YOU

Zero curvature representation graphically



$$v_1, v_2, v_3 \mapsto v_1', v_2', v_3'$$

Mapping in the "old" model

The "old" model: the algebra of observables is the Weyl one, $\mathbf{u}_j\mathbf{w}_j = q\mathbf{w}_j\mathbf{u}_j$. The automorphism is

$$\mathbf{w}_1' = \mathbf{w}_2\Lambda_3, \qquad \mathbf{w}_2' = \Lambda_3^{-1}\mathbf{w}_1, \quad \mathbf{w}_3' = \Lambda_2^{-1}\mathbf{u}_1^{-1},$$

$$\mathbf{u}_1' = \Lambda_2^{-1}\mathbf{w}_3^{-1}, \quad \mathbf{u}_2' = \Lambda_1^{-1}\mathbf{u}_3, \quad \mathbf{u}_3' = \mathbf{u}_2\Lambda_1$$

where

$$\begin{cases} \Lambda_1 = \mathbf{u}_1^{-1}\mathbf{u}_3 - q^{1/2}\mathbf{u}_1^{-1}\mathbf{w}_1 + \kappa_1\mathbf{w}_1\mathbf{u}_2^{-1} \\ \Lambda_2 = \dfrac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_3^{-1} + \dfrac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1}\mathbf{w}_2^{-1} - q^{-1/2}\dfrac{\kappa_1\kappa_3}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_2^{-1} \\ \Lambda_3 = \mathbf{w}_1\mathbf{w}_3^{-1} - q^{1/2}\mathbf{u}_3\mathbf{w}_3^{-1} + \kappa_3\mathbf{w}_2^{-1}\mathbf{u}_3 \end{cases}$$