Physics of Particles and Nuclei, Vol. 35, No. 5, 2004, pp. 1–31. Translated from Fizika Elementarnykh Chastits i Atomnogo Yadra, Vol. 35, No. 5, 2004, pp. @@–@@.

Quantum Integrable Models in Discrete 2 + 1-Dimensional Space–Time: Auxiliary Linear Problem on a Lattice, Zero-Curvature Representation, Isospectral Deformation of the Zamolodchikov–Bazhanov–Baxter Model

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Abstract—An invariant approach to quantum integrable models in wholly discrete 2 + 1-dimensional space–time is considered. An auxiliary linear problem on two-dimensional lattices generalizing quantum chains is formulated. A method of constructing a complete set of integrals of motion is given. For the two-dimensional lattices, we formulate and solve a zero-curvature representation allowing us to construct integrable evolutionary mappings. We place special emphasis on finite-dimensional representations of the algebra of observables, which exist if the Weyl algebra parameter is at a rational point of a unit circle (so-called "root of unity"). For this case, we derive a universal functional eigenvalue equation for integrals of motion. A groupoid of isospectral deformations is constructed for the finite-dimensional representations of the algebra of observables. Because the systems under consideration are finite-dimensional at the root of unity, the integrable systems can be treated as models of statistical mechanics on three-dimensional lattices. We formulate a method of constructing eigenstates of the models under consideration; the method is based on isospectral deformations (the method of quantum separation of variables for 2 + 1-dimensional models).

CONTENTS

1. INTRODUCTION

This paper is an attempt to systematize a series of new methods and results [1–9] concerning the construction and study of quantum evolution models, whose observables are defined on a two-dimensional lattice. Two-dimensional lattices are a discrete analogue of spacelike surfaces, while iterations of elementary evolutionary mappings of observables are treated as discrete time. According to the conventional terminology, such a construction belongs to models in a wholly discrete 2 + 1-dimensional space–time. The models to be formulated and studied in this paper allow a 1 + 1-dimensional space–time with fairly high internal symmetry ("isotopic symmetry") to be described in terms of quantum integrable models. It is worth noting that the space–time structure of 1 + 1 models is a one-dimensional, chain in contrast to a two-dimensional lattice of 2 + 1-dimensional models. The presence of the two formulations is related to an effect known as the "transmutation of the rank of the isotopic symmetry group into the dimension." Discrete 1 + 1-dimensional quantum models are studied by means of the quantum inverse-scattering method, a new branch of present-day mathematical physics. From the mathematical point of view, the method is a theory of quantum groups and their representations. Surprisingly, the 2 + 1-dimensional method of constructing and analyzing the integrable models to be described in this paper does not employ the technique of the quantum inverse-scattering problem method. From the point of view of the transmutation noted above, this is due to the fact that the 2 + 1-dimensional method is local on a two-dimensional lattice, while a vertex of the corresponding chain represents a strip on the two-dimensional lattice, i.e., the vertices are nonlocal objects. Because we will not employ the technique of the quantum inverse-scattering method, the integrability will be treated without the use of R matrices, monodromy matrices, etc.

As is well known, 1 + 1-dimensional integrable quantum models are often related to statistical mechanical models that are exactly solvable on two-dimensional lattices. Similarly, the analysis of statistical models exactly solvable on three-dimensional lattices (e.g., on a cubic lattice) is one of the applications of 2 + 1-dimensional quantum models. In order to distinguish quantum evolutionary models and models of statistical mechanics, we will refer to the spacelike lattice of 2 + 1-dimensional models as an auxiliary lattice.

1.1. Notations and Definitions

In this review, we use terminology for discrete quantum models that might not be widely used. Hence, it seems useful to define some notions in terms of the conventional quantum inverse-scattering method. We will deal with the physical interpretation of the integrability (Hopf algebra, comultiplication, etc.) rather than with mathematical aspects of it.

The formulation of a quantum method usually starts from the interlaced equation

$$\begin{aligned} &R_{v_1,v_2}(x/y)L_{v_1,V}(x)L_{v_2,V}(y) \\ &= L_{v_2,V}(y)L_{v_1,V}(x)R_{v_1,v_2}(x/y). \end{aligned} \tag{1}$$

Here, v_1 and v_2 are isomorphic n-dimensional vector spaces (usually, vector representations of a simple Lie algebra); V is a vector space, possibly not finite-dimensional; and

$$\begin{gathered} R_{v_1,v_2} \in \mathrm{End}(v_1 \otimes v_2), \quad L_{v_1,V} \in \mathrm{End}(v_1 \otimes V), \\ L_{v_2,V} \in \mathrm{End}(v_2 \otimes V). \end{gathered} \tag{2}$$

The arguments of the R matrix and L operators in Eq. (1) are referred to as spectral parameters. Therefore, the R matrix is a square $n^2 \times n^2$ matrix and the L operator is an $n \times n$ matrix whose elements are operators, namely, polynomials in the spectral parameters; the permutation relations for the matrix elements of L are defined by Eq. (1).

The integrable model is specified by a chain of the operators

$$\begin{aligned} &T_v(x) \\ &= L_{v,V_{m-1}}(x)\dots L_{v,V_1}(x)L_{v,V_0}(x) \in \mathrm{End}(v \otimes V^{\otimes m}), \end{aligned} \tag{3}$$

which is an ordered m product of different L operators and referred to as a monodromy matrix of the m-length chain. The numbers of V_μ spaces in Eq. (3) label the component in the direct product $V^{\otimes m}$. The monodromy matrices obey Eq. (1) because the right-hand side of Eq. (3) is an ordinary matrix product in the v space. The trace of the monodromy matrix, $t(x) = \mathrm{Trace}_v T_v(x) \in \mathrm{End}(V^{\otimes m})$, is referred to as a transfer matrix. The commutativity of transfer matrices, $t(x)t(y) = t(y)t(x)$, is a consequence of Eq. (1).

The V_μ spaces are referred to as quantum spaces, and operator-valued matrix elements of all L operators form an algebra of observables of the chain. This algebra is local since matrix elements of different L operators are commutative operators because of the specific structure of the direct product. The space v, which disappears when a transfer matrix has been constructed, is referred to as an auxiliary space; hence, the transfer matrix $t(x)$ can be referred to as an auxiliary transfer matrix. The coefficients of expansion of $t(x)$ in terms of the spectral parameter are polynomials in the algebra of observables, and because of the commutativity of transfer matrices, they form a set of commutative operators and are called integrals of motion.[1]

Within the framework of the quantum inverse-scattering method, the auxiliary linear problem is defined by the relations

$$\Phi_\mu L_{v,V_\mu}(x) = \Phi_{\mu+1}, \tag{4}$$

where $\Phi_\mu \in v \otimes V^{\otimes m}$, $\mu = 0, 1, \dots, m$. The monodromy matrix $T_v(x)$ is a monodromy of the vector Φ_0 circuiting a closed chain. It is noteworthy that the linear problem is not essentially used in quantum integrable models.

Together with intertwining equation (1) for L operators in the auxiliary space, there may exist the intertwining equation

$$\begin{aligned} &L_{v,V_1}(x)L_{v,V_2}(y)S_{V_1,V_2}(y/x) \\ &= S_{V_1,V_2}(y/x)L_{v,V_2}(y)L_{v,V_2}(x) \end{aligned} \tag{5}$$

in the quantum space, where S_{V_1,V_2} is a scalar in the space v. It is possible to express both the monodromy operator and the transfer matrix $Q(x)$ in terms of the operators S:

$$Q(x) = \mathrm{Trace}_V S_{V,V_{m-1}}(x)\dots S_{V,V_1}(x)S_{V,V_0}(x). \tag{6}$$

[1] A transfer matrix does not always generate a complete set of integrals of motion. In fact, in addition to the transfer matrix, one should consider all quantum characters of the monodromy matrix, with the trace being the first of them. The method of calculating the quantum characters is determined by the R matrix.

As follows from Eq. (5), $Q(x)t(y) = t(y)Q(x)$. When constructing Q, the quantum space serves as an auxiliary space; hence, $Q(x)$ is referred to as a quantum transfer matrix. It is worth noting that, when constructing the quantum transfer matrix, we use the trace over the quantum space supposed as a Hilbert space. In this case, under the extra assumption on the positive definiteness of the matrix elements, we can also consider statistical mechanics on a two-dimensional lattice. The statistical sum of a square $m \times k$ lattice is, by definition,

$$Z(x) = \mathrm{Trace}_{V^{\otimes m}} Q(x)^k. \tag{7}$$

If we do not require that the space would be a Hilbert space, we can use an evolution operator instead of the transfer matrix. The operator is defined as follows. Let us consider the auxiliary transfer matrices

$$\begin{aligned} t(x, y) = {} & \mathrm{Trace}_v L_{v, V_{m-1}}(x) L_{v, V_{m-2}}(xy) L_{v, V_{m-3}}(x) \\ & \times \ldots L_{v, V_1}(x) L_{v, V_0}(xy), \end{aligned} \tag{8}$$

where m is assumed to be even. They form a commutative set: $t(x, y)t(x', y) = t(x', y)t(x, y)$. We define the evolution operator

$$U(y) = S_{V_{m-1}, V_{m-2}}(y) S_{V_{m-3}, V_{m-4}}(y) \ldots S_{V_1, V_0}(y) U_0, \tag{9}$$

where the specific permutation operator U_0 is defined by the relations

$$\begin{aligned} e_\mu U_0 &= U_0 e_\mu, \quad \text{for odd } \mu, \\ \text{and} \quad e_\mu U_0 &= U_0 e_{\mu+3}, \quad \text{for even } \mu. \end{aligned} \tag{10}$$

Here, e_μ is an arbitrary element e of the μth component of the local algebra of observables:

$$e_\mu = 1 \otimes 1 \otimes \ldots \otimes \underbrace{e}_{\substack{\text{the } \mu\text{th} \\ \text{factor}}} \otimes \ldots, \tag{11}$$

where $\mu \in \mathbb{Z}_m$ if the length m of the chain is even. By definition, the operator $U(y)$ does not require the trace to be determined. Moreover, only the canonical transformation $e \mapsto SeS^{-1}$ rather than the operator S is needed to evaluate the one-step evolution $e \mapsto e' = U(y)eU(y)^{-1}$. According to Eqs. (5) and (10), after being expanded in a power series of x, the transfer matrix $t(x, y)$ generates the set of integrals of motion for the evolution operator $U(y)$:

$$t(x, y) = U(y)t(x, y)U(y)^{-1}. \tag{12}$$

It is worth noting that statistical sum (7) for the $m \times m$ lattice can be defined in terms of the evolution operator: $Z(y) = \mathrm{Trace}\, U(y)^m$.

1.2. 2 + 1-Dimensional Models

The problem of integrable models with higher dimensions virtually reduces to the extension of the notion of a one-dimensional chain to the two-dimensional case, namely, to an auxiliary lattice. The method proposed in this paper is based on a specific auxiliary linear problem different from (4) and dependent on local properties of the auxiliary lattice. In this case, no analogues of the L operator and auxiliary space are employed; moreover, basic notions of quantum groups, namely, fundamental R matrices and relations similar to (1), need not be used. However, we will use all other notions of quantum groups: the local algebra of observables; an analogue, defined as a determinant, of the auxiliary transfer matrix; quantum intertwiners, i.e., analogues of S operators satisfying the tetrahedron equation; and evolution operators and quantum transfer matrices, which are used for treating statistical mechanics on a cubic lattice.

It is important that the shape, as well as the size, of the auxiliary lattice for 2 + 1-dimensional models serve as parameters. Sometimes, the inverse transition from the 2 + 1-dimensional local formalism to a 1 + 1-dimensional one (i.e., transmutation of the dimension into the rank) is also possible. In this case, the spatial dimensions, n and m, turn into the "dimension of isotopic symmetry group" and the chain length, respectively. In this way, depending on the shape of the auxiliary lattice, various 1 + 1-dimensional models can be obtained, for example, the models associated with $\mathcal{U}_q(\widehat{sl_n})$, the quantum relativistic Toda chain and a series of its generalizations, the quantum discrete Liouville model and its generalizations, etc.

2. FORMULATION OF THE MODEL

2.1. Auxiliary Lattice

First of all, we introduce the notion of a two-dimensional auxiliary lattice, which replaces that of a one-dimensional chain.

Definition 1. *A lattice formed by directed lines on a torus will be referred to as an auxiliary lattice. The lines may intersect in pairs or be nonintersecting. We also require that there be no lines a trivial homotopy class and no topologically trivial entanglements. The key requirement is that the number Δ of vertices should be equal to the number of sites, i.e., the lattice cannot be mapped on a sphere.*

In the construction proposed, we will use vertices of the auxiliary lattice and its sites. A pair $\mathbf{u}_V$ and $\mathbf{w}_V$ of invertible generators of a simple Weyl algebra with a unique parameter q for the whole lattice and, in addition, a $\mathbb{C}$-number parameter κ_V are assigned to each vertex V. The algebra of observables, defined by the equalities

$$\begin{aligned} \mathbf{u}_V \mathbf{u}_{V'} = \mathbf{u}_{V'} \mathbf{u}_V, \quad & \mathbf{w}_V \mathbf{w}_{V'} = \mathbf{w}_{V'} \mathbf{w}_V, \\ \mathbf{u}_V \mathbf{w}_{V'} &= q^{\delta_{V, V'}} \mathbf{w}_{V'} \mathbf{u}_V \end{aligned} \tag{13}$$

is a local Weyl algebra.

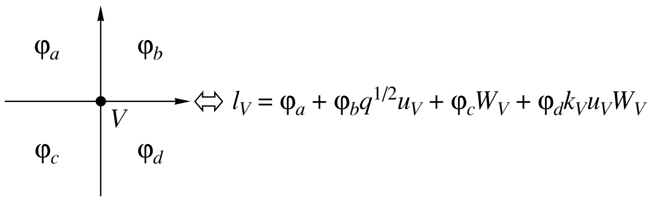

Fig. 1. *V*th vertex of an auxiliary lattice.

For each site S of the lattice, we define an auxiliary element φ_S belonging to the formal left module of the local Weyl algebra.

Each vertex of the lattice is surrounded by four sites (see Fig. 1). The sites surrounding a vertex V and the auxiliary linear elements are denoted by (a, b, c, d) and by $(\varphi_a, \ldots, \varphi_d)$, respectively.

Instead of the notion of L operator, we introduce the definition of a vertex linear form.

Definition 2. *For each vertex V, we define a vertex linear form l_V as a linear superposition of site elements φ with coefficients from the vertex Weyl algebra*:

$$l_V \stackrel{\text{def}}{=} \varphi_a + \varphi_b q^{1/2}\mathbf{u}_V + \varphi_c \mathbf{w}_V + \varphi_d \kappa_V \mathbf{u}_V \mathbf{w}_V. \tag{14}$$

In fact, this definition of the vertex linear form is a generalization of auxiliary linear problem (4) in the form $l_V = 0$.

It is noteworthy that the definition of l_V made above involves directed lines. Hence, the rule given in Fig. 1 should be applied to all the vertices of the lattice, because the auxiliary lattice, by definition, is formed by directed lines. This implies the mapping of the torus onto a plane parallelogram with identified sides. Let us fix one of the possible mappings of a lattice. In this case, some boundary sites will be divided into several parts. The parts on opposite sides of the parallelogram must be identified. However, it is necessary to slightly change the definition of the auxiliary variables φ_S. Let S be one such cut site and φ_S be an auxiliary variable assigned to one of its pieces. In this case, the auxiliary variables $x\varphi_S$ and $y\varphi_S$ (x and y are complex numbers) are assigned to the cut-site pieces obtained by passing the first and second cycles of the torus, respectively. Such conditions are referred to as quasiperiodic boundary conditions, and x and y are $\mathbb{C}$-number monodromies. A lattice formed by two lines, two vertices, and two sites is shown in Fig. 2 as an example. The horizontal and vertical directions correspond to the first and second cycles of the torus, respectively. Site 1 is thus cut into four parts, while site 2 is cut into three parts. If we go from the left (upper) side to the right (lower) side of the fundamental rectangular of the torus, then the variable φ_S is multiplied by x (y). It is important that there are just two monodromies, which correspond to the two independent cycles of the torus.

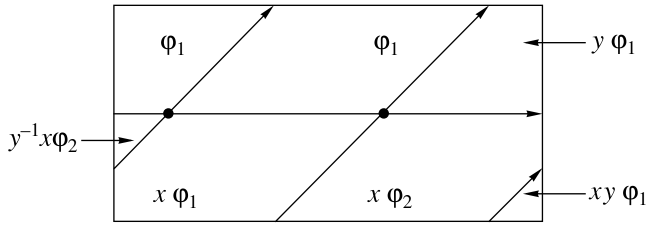

Fig. 2. An example of quasi-periodic boundary conditions.

2.2. Generating Functional of the Integrals of Motion

As a result of the rules defined above, we obtain the system of linear relations

$$l_V = \sum_S \varphi_S \mathbf{L}_{S|V}(x, y) \tag{15}$$

for each lattice on the torus. Here, $\mathbf{L}_{S|V}$ is the matrix of coefficients of linear equations (14), which is defined with allowance for the quasiperiodic boundary conditions described above. Since the algebra of observables is local, we can introduce the operator-valued determinant

$$\mathbf{j}(x, y) = \det\|\mathbf{L}_{S|V}(x, y)\| \tag{16}$$

of the matrix of coefficients. This quantity is well defined. Indeed, according to our definition of the auxiliary lattice, the number of vertices is equal to the number of sites, so that $\|\mathbf{L}_{S|V}\|$ is a square $\Delta \times \Delta$ matrix. Secondly, according to linear relation (14), the operator-valued matrix elements of $\mathbf{L}_{S|V}$ are locally commutative:

$$\mathbf{L}_{S|V}\mathbf{L}_{S'|V'} = \mathbf{L}_{S'|V'}\mathbf{L}_{S|V}, \quad \text{if} \quad V \neq V'. \tag{17}$$

Therefore, the ordering need not be performed and this determinant is given by the conventional formula

$$\det\|\mathbf{L}_{S|V}\| = \sum_{\text{permutations of } \sigma} \prod_V \mathbf{L}_{S = \sigma(V)|V}. \tag{18}$$

The determinant $\mathbf{j}(x, y)$ is a Laurent polynomial in x and y:

$$\mathbf{j}(x, y) = \sum_{\alpha, \beta \in \Sigma} x^\alpha y^\beta \mathbf{j}'_{\alpha, \beta}. \tag{19}$$

The summation in Eq. (19) is taken over a two-dimensional discrete manifold Σ, called a Newton polygon, for the polynomial $\mathbf{j}(x, y)$. The numbers α and β in Eq. (19) are always integers. Both the type and shape of Σ depend on the dimension of an auxiliary lattice and its shape and on the arrangement of the parameters x and y in the definition of φ_S.

In any case, let $\mathcal{N} = \mathbf{j}'_{0,0}$ and

$$\mathbf{j}_{\alpha, \beta} = \mathbf{j}'_{\alpha, \beta}\mathcal{N}^{-1}, \tag{20}$$

so that $\mathbf{j}_{0,\,0} \equiv 1$. The following statement is not a theorem since we have proved it only for specific regular auxiliary lattices (e.g., for square or spiral lattices). However, we have analytically verified this statement for a series of finite lattices.

Statement 1. *For arbitrary (in shape and size) lattices, the matrix elements* $\mathbf{j}_{\alpha,\,\beta}$ *satisfy the permutation relations*

$$\mathbf{j}_{\alpha,\,\beta}\mathbf{j}_{\alpha',\,\beta'} = q^{\pm(\alpha\beta'-\beta\alpha')}\mathbf{j}_{\alpha',\,\beta'}\mathbf{j}_{\alpha,\,\beta}. \tag{21}$$

(Relation (21) remains valid for specific lattices not considered here, namely, for lattices with lines of zero homotopy class or with lines having entanglements and self-intersections.) Relation (21) holds for any choice of the parts of sites to which the variables φ_S *without factors x and y are assigned. The sign of the exponent in Eq. (21) is the same for all pairs* (α, β) *and* (α', β') *and depends on the choice of directions of the two nonequivalent cycles corresponding to the monodromies x and y.*

It is worth noting that the arbitrary choice of the cycles is a trivial consequence of the definition of a determinant.

Statement *1* is an assertion concerning integrability. According to relation (21), the elements

$$\mathbf{t}_{\alpha,\,\beta} \overset{\text{def}}{=} \mathbf{j}_{\alpha,\,\beta}\mathbf{j}_{1,\,0}^{-\alpha}\mathbf{j}_{0,\,1}^{-\beta} \tag{22}$$

form a commutative set; this is a criterion for integrability. The elements $\mathbf{t}_{\alpha,\,\beta}$ are constructed as rational functions on the algebra of observables. Moreover, as is proved in what follows, they can always be reduced to polynomials. There is another assertion in addition to Statement *1*.

Statement 2. *In the cases under consideration, the number of (algebraically) independent elements* $\mathbf{t}_{\alpha,\,\beta}(\alpha, \beta) \in \Sigma$ *is equal to* $\Delta - 1$, *provided that the auxiliary lattice has no lines of trivial homotopy class and links. Here,* Δ *is the number of vertices in the lattice. The* Δ*th commutative element is an arbitrary function of* $\mathbf{j}_{1,\,0}$ *and* $\mathbf{j}_{0,\,1}$.

The proof of Statement *2* concerning a complete set of integrals of motion will be outlined below.

The presented method of constructing a complete set of commutative polynomials on the algebra of observables is a basis of the integrable model. The method replaces the construction of auxiliary monodromy matrices and the evaluation of their quantum characters. It is remarkable that we did not need to employ interlaced equations (1), a basis of the 1 + 1-dimensional quantum inverse-scattering method. However, when formulating three-dimensional models in terms of two-dimensional ones, *L* operators of the quantum inverse-scattering method originate automatically.

Furthermore, an essential difference between $t(x)$ and $\mathbf{j}(x, y)$ is that the set $\mathbf{j}(x, y)$ contains a noncommutative pair, $\mathbf{j}_{1,\,0}$ and $\mathbf{l}_{0,\,1}$, which serves as a canonical pair for the “center of inertia.”

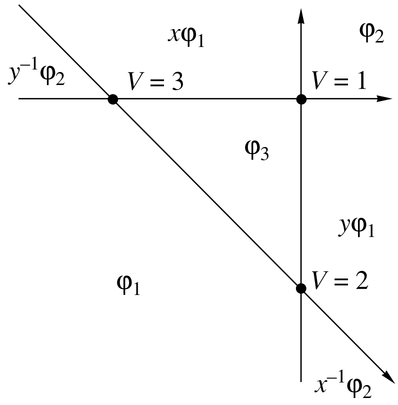

Fig. 3. Oriented triangle on a torus.

2.3. Examples

We now present some examples of auxiliary lattices and determinants $\mathbf{j}(x, y)$.

2.3.1. Triangle. Let us consider a torus with an oriented triangle shown in Fig. 3, where the indexing of vertices and sites and the arrangement of spectral parameters *x* and *y* are also given: the passage from the lower (left) edge to the upper (right) corresponds to the multiplication by *x* (*y*). This triangle is the so-called minimum-size kagome lattice.

According to the rules given in Fig. 1, we have the three linear forms

$$\begin{aligned} l_1 &= x\varphi_1 + \varphi_2 q^{1/2}\mathbf{u}_1 + \varphi_3\mathbf{w}_1 + y\varphi_1\kappa_1\mathbf{u}_1\mathbf{w}_1, \\ l_2 &= \varphi_3 + y\varphi_1 q^{1/2}\mathbf{u}_2 + \varphi_1\mathbf{w}_2 + x^{-1}\varphi_2\kappa_2\mathbf{u}_2\mathbf{w}_2, \\ l_3 &= x\varphi_1 + \varphi_3 q^{1/2}\mathbf{u}_3 + y^{-1}\varphi_2\mathbf{w}_3 + \varphi_1\kappa_3\mathbf{u}_3\mathbf{w}_3, \end{aligned} \tag{23}$$

which are defined by the following matrix of coefficients:

$$\mathbf{L} = \begin{pmatrix} x + y\kappa_1\mathbf{u}_1\mathbf{w}_1 & \mathbf{w}_2 + yq^{1/2}\mathbf{u}_2 & x + \kappa_3\mathbf{u}_3\mathbf{w}_3 \\ q^{1/2}\mathbf{u}_1 & x^{-1}\kappa_2\mathbf{u}_2\mathbf{w}_2 & y^{-1}\mathbf{w}_3 \\ \mathbf{w}_1 & 1 & q^{1/2}\mathbf{u}_3 \end{pmatrix}. \tag{24}$$

The determinant of $\mathbf{L}$ is

$$\begin{aligned} \det\mathbf{L} &= xq^{1/2}\mathbf{u}_1 + y^{-1}\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3 \\ &+ x^{-1}yq^{1/2}\kappa_1\kappa_2\mathbf{u}_1\mathbf{w}_1\mathbf{u}_2\mathbf{w}_2\mathbf{u}_3 - x^{-1}\kappa_2\kappa_3\mathbf{w}_1\mathbf{u}_2\mathbf{w}_2\mathbf{u}_3\mathbf{w}_3 \\ &- yq^{3/2}\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 - xy^{-1}\mathbf{w}_3 + \mathbf{H}q^{1/2}\mathbf{u}_2\mathbf{u}_3\mathbf{w}_3, \end{aligned} \tag{25}$$

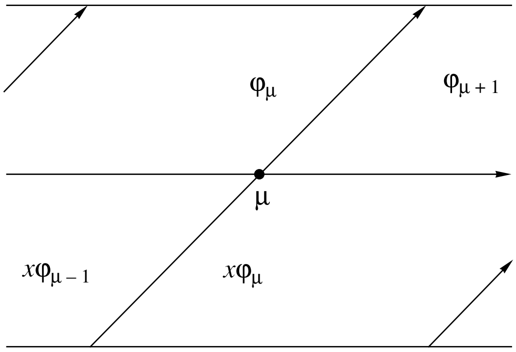

Fig. 4. Fragment of a spiral lattice.

where

$$\begin{aligned}\mathbf{H} &= (\mathbf{w} + \kappa_2\mathbf{u}^{-1} - q^{1/2}\kappa_2\mathbf{u}^{-1}\mathbf{w}) \\ &+ \mathbf{s}(\mathbf{w}^{-1} + \kappa_1\mathbf{u} - q^{1/2}\kappa_3\mathbf{u}\mathbf{w}^{-1}),\end{aligned} \tag{26}$$

$$\begin{aligned}\mathbf{u} &= \mathbf{w}_2^{-1}\mathbf{w}_3, \quad \mathbf{w} = \mathbf{w}_1\mathbf{u}_3^{-1}, \\ \mathbf{s} &= -q^{-1/2}\mathbf{u}_1\mathbf{w}_1\mathbf{u}_2^{-1}\mathbf{w}_2\mathbf{u}_3^{-1}\mathbf{w}_3^{-1}.\end{aligned} \tag{27}$$

Using a scheme based on Eq. (22), we conclude that the operators $\mathbf{u}_2\mathbf{u}_3$ and $\mathbf{u}_1^{-1}\mathbf{w}_3$ can be chosen as a noncommutative pair and that the polynomials $\mathbf{H}$ and $\mathbf{s}$ given by Eqs. (26) and (27) and an arbitrary polynomial $f(\mathbf{u}_2\mathbf{u}_3, \mathbf{u}_1^{-1}\mathbf{w}_3)$ form a complete set of commutative elements.

2.3.2. Spiral. One more example of the auxiliary lattice is an arbitrary spiral lattice. Namely, let $\mathcal{A}$ and $\mathcal{B}$ be two basic cycles on a torus. The spiral with m turns is a line of homotopy class $\mathcal{A} + m\mathcal{B}$. The second line is a line of class $\mathcal{A}$. A spiral lattice with $m = 2$ is presented in Fig. 2. Figure 4 shows one fragment containing the μth vertex of the lattice ($\mu = 0, \ldots, m-1$). The boundary conditions in the vertical direction (cycle $\mathcal{B}$, parameter x) are shown in Fig. 4, while those in the horizontal direction (cycle $\mathcal{A}$) have the form $x\varphi_{-1} = xy^{-1}\varphi_{m-1}$ and $\varphi_m = y\varphi_0$.

In order to evaluate the determinant, we rewrite the linear problem illustrated in Fig. 1 in a special way as follows. For a system of linear equations $l_V = \sum_S \varphi_S \mathbf{L}_{S|V}$, the determinant of its matrix of coefficients corresponding to the homogeneous system $l_V = 0$ has the form $\varphi_S \det\|\mathbf{L}\| = 0$. In the next section, we discuss this aspect in detail. The linear equation

$$\begin{aligned}[l_\mu =]\varphi_\mu + \varphi_{\mu+1}q^{1/2}\mathbf{u}_\mu + x\varphi_{\mu-1}\mathbf{w}_\mu \\ + x\varphi_\mu\kappa_\mu\mathbf{u}_\mu\mathbf{w}_\mu = 0,\end{aligned} \tag{28}$$

corresponding to the μth vertex shown in Fig. 4, can be rewritten in the form of Eq. (4):

$$\Phi_\mu L_\mu(x) = \Phi_{\mu+1}(-q^{1/2}\mathbf{u}_\mu). \tag{29}$$

Here, the boundary condition $\Phi_m = y\Phi_0$ is imposed on the vector rows $\Phi_\mu = (\varphi_\mu, \varphi_{\mu-1})$, and the matrix L is

$$L_\mu(x) = \begin{pmatrix} 1 + x\kappa_\mu\mathbf{u}_\mu\mathbf{w}_\mu & -q^{1/2}\mathbf{u}_\mu \\ x\mathbf{w}_\mu & 0 \end{pmatrix}. \tag{30}$$

The matrix L is a Lax operator for the quantum relativistic Toda chain. Linear equations (28) can be written out as the system of two linear equations

$$\Phi_0 \cdot \left(T(x) - y\prod_{\mu=0}^{m-1}(-q^{1/2}\mathbf{u}_\mu) \right) = 0, \tag{31}$$

where the monodromy matrix is

$$T(x) = L_0(x)L_1(x)\ldots L_{m-1}(x). \tag{32}$$

It is evident that the determinant of $\mathbf{j}(x, y)$ is proportional to a properly defined determinant of the matrix $T(x) - y\prod_{\mu=0}^{m-1}(-q^{1/2}\mathbf{u}_\mu)$. Using combinatorial analysis, one can rigorously prove that

$$\mathbf{j}(x, y) = t(x) - y\prod_{\mu=0}^{m-1}(-q^{1/2}\mathbf{u}_\mu) - x^m y^{-1}\prod_{\mu=0}^{m-1}(-\mathbf{w}_\mu), \tag{33}$$

where the trace $t(x)$ of the monodromy matrix generates a commutative set of integrals of motion: $t(x) = \sum_{k=0}^m x^k \mathbf{t}_k$. Since $\mathbf{t}_0 = 1$, the determinant need not be normalized. For example, $\mathbf{t}_m = \prod_{\mu=0}^{m-1}\kappa_\mu\mathbf{u}_\mu\mathbf{w}_\mu$ and $\mathbf{t}_1 = \sum_{\mu\in\mathbb{Z}_m}\kappa_\mu\mathbf{u}_\mu\mathbf{w}_\mu - q^{1/2}\mathbf{u}_\mu\mathbf{w}_{\mu+1}$. It is easy to prove that $\mathbf{t}_k$ is a polynomial of degree k in both $\mathbf{w}_\mu$ and $\mathbf{u}_\mu$. The set $\mathbf{t}_k$ is complete and implicitly contains the first element of the noncommutative pair, with its second element being $\prod_\mu \mathbf{u}_\mu$. Hence, relation (21) for the Toda chain has the form

$$\left(\prod_\mu \mathbf{u}_\mu\right) t(x) = t(qx)\left(\prod_\mu \mathbf{u}_\mu\right). \tag{34}$$

2.3.3. Square lattice. A square lattice is our basic example because it can have arbitrary dimensions in both directions, i.e., it is the simplest, actually two-dimensional, lattice. Vertices of a square lattice are numbered by pairs $V = (n_2, n_3)$, where $n_2 \in \mathbb{Z}_{N_2}$, $n_3 \in \mathbb{Z}_{N_3}$, and N_2 and N_3 are the dimensions of the torus.[2]

[2] Looking ahead, indices n1 and N1 will be used below for discrete time of quantum mechanics and for the third spatial coordinate of statistical mechanics.

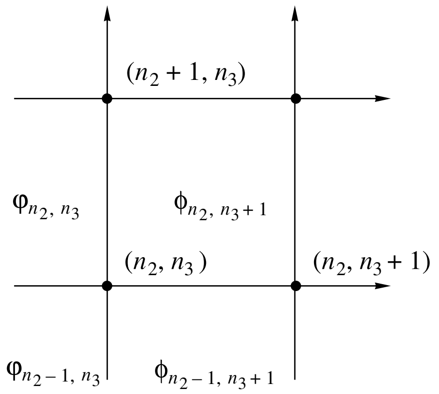

Fig. 5. Fragment of an auxiliary square lattice.

The linear relation for an (n_2, n_3) pair takes the form

$$\begin{aligned} l_{n_2, n_3} &= \varphi_{n_2, n_3} + \varphi_{n_2, n_3+1} q^{1/2} \mathbf{u}_{n_2, n_3} \\ &+ \varphi_{n_2-1, n_3} \mathbf{w}_{n_2, n_3} + \varphi_{n_2-1, n_3+1} \kappa_{n_2, n_3} \mathbf{u}_{n_2, n_3} \mathbf{w}_{n_2, n_3}. \end{aligned} \tag{35}$$

Here, $n_2 = 0, \ldots, N_2 - 1$; $n_3 = 0, \ldots, N_3 - 1$; and the periodic boundary conditions

$$\varphi_{-1, n_3} = x \varphi_{N_2-1, n_3}, \quad \varphi_{n_2, N_3} = y \varphi_{n_2, 0}, \tag{36}$$

are imposed on the linear variables. According to the definition of the matrix $\mathbf{L}$, $l_V = \sum_S \varphi_S \mathbf{L}_{S|V}$. As follows from Eqs. (35) and (36),

$$\begin{aligned} \mathbf{L}_{n_2, n_3 | n_2, n_3} &= 1, \\ \mathbf{L}_{n_2, n_3+1 | n_2, n_3} &= q^{1/2} \mathbf{u}_{n_2, n_3} y^{\delta_{n_3, N_3-1}}, \\ \mathbf{L}_{n_2-1, n_3 | n_2, n_3} &= \mathbf{w}_{n_2, n_3} x^{\delta_{n_2, 0}}, \\ \mathbf{L}_{n_2-1, n_3+1 | n_2, n_3} &= \kappa_{n_2, n_3} \mathbf{u}_{n_2, n_3} \mathbf{w}_{n_2, n_3} x^{\delta_{n_2, 0}} y^{\delta_{n_3, N_3-1}}. \end{aligned} \tag{37}$$

Here, $n_2 \in \mathbb{Z}_{N_2}$, $n_3 \in \mathbb{Z}_{N_3}$, and all the remaining $\mathbf{L}_{S|V} = 0$. The indexing used is shown in Fig. 5. The determinant need not be normalized because $\mathbf{j}_{0, 0} \equiv 1$ in the expansion

$$\mathbf{j}(x, y) = \sum_{\nu_2 = 0}^{N_2} \sum_{\nu_3 = 0}^{N_3} y^{\nu_2} x^{\nu_3} \mathbf{j}_{\nu_2, \nu_3}. \tag{38}$$

Let

$$\mathbf{U}_{n_2} = \prod_{n_3 = 0}^{N_3 - 1} (-q^{1/2} \mathbf{u}_{n_2, n_3}), \quad \mathbf{W}_{n_3} = \prod_{n_2 = 0}^{N_2 - 1} (-\mathbf{w}_{n_2, n_3}). \tag{39}$$

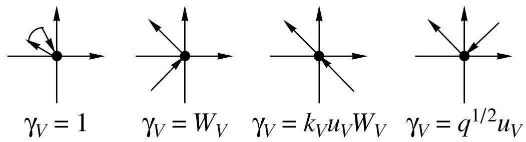

Fig. 6. Four variants of a path through a vertex on the square lattice.

It is easy to prove that

$$\begin{aligned} \sum_{\nu_2 = 0}^{N_2} y^{\nu_2} \mathbf{j}_{\nu_2, 0} &= \prod_{n_2 = 0}^{N_2 - 1} (1 - y \mathbf{U}_{n_2}), \\ \sum_{\nu_3 = 0}^{N_3} x^{\nu_3} \mathbf{j}_{0, \nu_3} &= \prod_{n_3 = 0}^{N_3 - 1} (1 - x \mathbf{W}_{n_3}). \end{aligned} \tag{40}$$

Without loss of generality, $\mathbf{U}_0$ and $\mathbf{W}_0$ can be chosen as a noncommutative pair so that the set of the commutative integrals of motion

$$\mathbf{t}_{\nu_2, \nu_3} = \mathbf{j}_{\nu_2, \nu_3} \mathbf{U}_0^{-\nu_2} \mathbf{W}_0^{-\nu_3} \tag{41}$$

contains all the remaining $\mathbf{U}_{n_2} \mathbf{U}_0^{-1}$ and $\mathbf{W}_{n_3} \mathbf{W}_0^{-1}$. It is convenient to write the permutation relations between $\mathbf{j}_{\nu_2, \nu_3}$ and $(\mathbf{U}_{n_2}, \mathbf{W}_{n_3})$ in the form

$$\begin{aligned} \mathbf{j}(x, y) \mathbf{U}_{n_2} &= \mathbf{U}_{n_2} \mathbf{j}(q^{-1} x, y), \\ \mathbf{j}(x, y) \mathbf{W}_{n_3} &= \mathbf{W}_{n_3} \mathbf{j}(x, q y). \end{aligned} \tag{42}$$

2.3.3.1. Combinatorial representation. There exists a remarkable combinatorial representation for $\mathbf{j}_{\nu_2, \nu_3}$ on a square lattice. The representation is defined with the help of specific ways on the lattice. Each way passes through each vertex and each site just once. Figure 6 shows the four possible ways passing through a vertex. In general, these ways are multiply connected; an elementary connected component is shown on the left side of Fig. 6. The torus cycles $\mathcal{B}$ and $\mathcal{A}$ are oriented upwards and from right to left, respectively. Each way $\mathcal{W}$ belongs to a certain homotopy class $c(\mathcal{W}) = \nu_2 \mathcal{A} + \nu_3 \mathcal{B}$. Let γ_V be the factor associated with a variant of the passage through a vertex. These factors are written out in Fig. 6. Then,

$$\mathbf{j}_{\nu_2, \nu_3} = \sum_{\mathcal{W} : c(\mathcal{W}) = \nu_2 \mathcal{A} + \nu_3 \mathcal{B}} (-)^{nm + \#\mathrm{loops}} \prod_{\text{along } \mathcal{W}} \gamma_V. \tag{43}$$

This representation coincides with the definition of a determinant as a sum over permutations σ: $\det \mathbf{L} = \sum_\sigma (-)^\sigma \prod_j L_{j, \sigma(j)}$. In the case of nonsquare lattices, the variants of the passage through a vertex and the corresponding factors γ_V depend on the type of vertex.

2.3.3.2. Lax operator. In much the same was as for the spiral lattice considered above, the homogeneous linear problem $l_{n_2, n_3} = 0$ can be expressed in terms of vector rows and Lax operators. For a fixed n_3 and all $n_2 = 0, \ldots, N_2 - 1$, system (35) can be rewritten in the form

$$\Phi^{(n_3)} \cdot A^{(n_3)}(x) = -\Phi^{(n_3+1)} B^{(n_3)}(x), \tag{44}$$

where $\Phi^{(n_3)} = (\varphi_{0, n_3}, \varphi_{1, n_3}, \ldots, \varphi_{N_2-1, n_3})$ and

$$A^{(n_3)}(x) = \sum_{n_2 \in \mathbb{Z}_{N_2}} (e_{n_2, n_2} + x^{\delta_{n_2, 0}} e_{n_2+1, n_2} \mathbf{w}_{n_2, n_3}),$$

$$B^{(n_3)}(x) \tag{45}$$

$$= \sum_{n_2 \in \mathbb{Z}_{N_2}} (e_{n_2, n_2} q^{1/2} \mathbf{u}_{n_2, n_3} + x^{\delta_{n_2, 0}} \kappa_{n_2, n_3} \mathbf{u}_{n_2, n_3} \mathbf{w}_{n_2, n_3}).$$

Here, e_{n_2', n_3} is the identity (n_2', n_2) matrix in $\mathrm{End}(\mathbb{C}^{N_2})$. Introducing the matrices

$$L^{(n_3)}(x) = A^{(n_3)}(x)(B^{(n_3)}(x))^{-1}(-q^{1/2}\mathbf{u}_{0, n_3})^{-1}, \tag{46}$$

we rewrite the system of homogeneous equations along the horizontal strip shown in Fig. 5 in the form of Eq. (4). According to the results of [10], matrices (46) satisfy interlaced equation (1) for a certain special R matrix associated with $\mathcal{U}_q(\widehat{sl_{N_2}})$. The factor $\mathbf{u}_{0, n_3}^{-1}$ is needed for the transfer matrix constructed from monodromy (46) to generate the commutative (but not complete) set $\mathrm{t}\mathbf{t}_{1, \nu_3} \mathbf{W}_0^{\nu_3} = \mathbf{j}_{1, \nu_3} \mathbf{U}_0^{-1}$.

2.4. Properties of Inverse Matrix of Coefficients

The matrices $\mathbf{L}$, in addition to remarkable properties of its determinant $\mathbf{j} = \det\|\mathbf{L}_{S|V}\|$, have other, no less wonderful properties. The rigorous proofs of the results presented in this section can be found in [5].

Since elements of different columns of matrix $\mathbf{L}$ are commuting operators, its algebraic complements, as well as its determinant, are well defined. Let $\mathbf{A}_{V|S}$ be the algebraic complement of a matrix element $\mathbf{L}_{S|V}$, so that

$$\sum_V \mathbf{L}_{S|V} \mathbf{A}_{V|S'} = \mathbf{j}\delta_{S|S'}, \tag{47}$$

where $\mathbf{j} = \det\mathbf{L}$. Hence, the inverse matrix

$$(\mathbf{L}^{-1})_{V|S} = \mathbf{A}_{V|S}\mathbf{j}^{-1} \tag{48}$$

is well defined. The commutativity of all elements of any row,

$$(\mathbf{L}^{-1})_{V|S}(\mathbf{L}^{-1})_{V|S'} = (\mathbf{L}^{-1})_{V|S'}(\mathbf{L}^{-1})_{V|S} \quad \forall V, S, S', \tag{49}$$

is the basic feature of matrix $\mathbf{L}^{-1}$. Instead of reading paper [5], an interested reader could verify the validity of formula (49) for simple matrix (24).

We choose V_0 and S_0 from all V and S such that $\mathbf{A}_{V_0|S_0} \neq 0$ and define the normalized operator-valued coefficients

$$\mathbf{m}_{S, S_0}(V_0) = (\mathbf{L}^{-1})^{-1}_{V_0|S_0}(\mathbf{L}^{-1})_{V_0|S} = \mathbf{A}_{V_0|S}\mathbf{A}^{-1}_{V_0|S_0}. \tag{50}$$

It follows from (49) that

$$\mathbf{m}_{S, S_0}(V_0), \mathbf{m}_{S', S_0}(V_0) = \mathbf{m}_{S', S_0}(V_0)\mathbf{m}_{S, S_0}(V_0) \tag{51}$$

and

$$\begin{aligned} \mathbf{m}_{S, S_0}(V_0)\mathbf{j} &= \mathbf{j}\tilde{\mathbf{m}}_{S, S_0}(V_0), \\ \tilde{\mathbf{m}}_{S, S_0}(V_0) &= \mathbf{A}^{-1}_{V_0|S_0}\mathbf{A}_{V_0|S}. \end{aligned} \tag{52}$$

These formulas yield the following method of solving the system of homogeneous equations $l_V = 0$, $\forall V$. Let us choose V_0 and S_0, with $\mathbf{A}_{V_0|S_0} \neq 0$. The incomplete system $l_V = 0$, $V \neq V_0$, has the solution

$$\varphi_S = \varphi_{S_0}\mathbf{m}_{S, S_0}, \tag{53}$$

where all $\mathbf{m}$ are commuting operators. With regard to (53), the remaining equation $l_{V_0} = 0$ reduces to

$$\varphi_{S_0}\mathbf{j} = 0. \tag{54}$$

According to (52), this condition is satisfied for any site,

$$\varphi_S\mathbf{j} = 0 \quad \forall S. \tag{55}$$

The results of this section can be obtained by using only the locality of the coefficients $\mathbf{L}_{S|V}$: $\mathbf{L}_{S|V}\mathbf{L}_{S'|V'} = \mathbf{L}_{S'|V'}\mathbf{L}_{S|V}$ $\forall S, S'$ for $V \neq V'$. As concerns a local Weyl algebra, it should be noted that, for any Δ-vertex lattice under consideration, the number $\Delta - 1$ of Weyl pairs, after taking away the V_0th vertex, is equal to the number of independent commutative operators $\mathbf{m}_{S, S_0}(V_0)(x, y)$.

3. FINITE-DIMENSIONAL REPRESENTATIONS OF THE ALGEBRA OF OBSERVABLES

3.1. Finite-Dimensional Representation of the Weyl Algebra at the Root of Unity

3.1.1. Simple Weyl algebra at the root of unity. We now consider finite-dimensional representations of the Weyl algebra, which exist if the Weyl parameter q is a root of unity:

$$q = e^{2i\pi/N}, \quad q^{1/2} = e^{i\pi/N}, \tag{56}$$

where N is an arbitrary natural number greater than unity. The Weyl pair $(\mathbf{u}, \mathbf{w})$ allows the finite-dimensional unitary representation

$$\mathbf{u} = u\mathbf{x}, \quad \mathbf{w} = w\mathbf{z}. \tag{57}$$

Here, u and $w \in \mathbb{C}$ are parameters, while $\mathbf{x}$ and $\mathbf{z}$ can be defined in the basis $|\sigma\rangle = |\sigma \bmod N\rangle$, $\langle\sigma|\sigma'\rangle = \delta_{\sigma,\sigma'}$, as

$$\langle\sigma|\mathbf{x}|\sigma'\rangle = q^{\sigma}\delta_{\sigma,\sigma'}, \quad \langle\sigma|\mathbf{z}|\sigma'\rangle = \delta_{\sigma,\sigma'+1}. \tag{58}$$

The Nth powers of $\mathbf{u}$ and $\mathbf{w}$ are numbers: $\mathbf{u}^N = u^N$ and $\mathbf{w}^N = w^N$. In addition to the unitary finite-dimensional representation, there exists an infinite-dimensional real representation at the root of unity.

3.1.2. Algebra of observables at the root of unity. To extend this representation to an entire lattice with Δ vertices, we define

$$\mathbf{u}_V = u_V\mathbf{x}_V, \quad \mathbf{w}_V = w_V\mathbf{z}_V. \tag{59}$$

Here, the set of 2Δ parameters u_V and w_V is, in general, arbitrary, and

$$\begin{aligned}\mathbf{x}_V &= 1\otimes 1\otimes\ldots\otimes \underbrace{\mathbf{x}}_{\substack{\text{the } V\text{th}\\ \text{factor}}} \otimes\ldots,\\ \mathbf{z}_V &= 1\otimes 1\otimes\ldots\otimes \underbrace{\mathbf{z}}_{\substack{\text{the } V\text{th}\\ \text{factor}}} \otimes\ldots.\end{aligned} \tag{60}$$

Therefore, the N^{Δ}-dimensional Hilbert space $\mathcal{H}$ with the basis

$$|\sigma\rangle = |\sigma_1\rangle\otimes|\sigma_2\rangle\otimes\ldots\otimes|\sigma_V\rangle\otimes\ldots \tag{61}$$

is defined for the algebra of observables. The auxiliary elements φ_S belong to the dual space $\mathcal{H}^*$. In what follows, we use the Dirac notation $\langle\varphi_S|$. In the specific case $N = 2$, the matrices

$$\begin{aligned}&\mathbf{x} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\\ &-q^{1/2}\mathbf{x}\mathbf{z} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\end{aligned} \tag{62}$$

are the Hermitian Pauli matrices. It is easy to verify that, for any lattice, all operators $\mathbf{j}_{\alpha,\beta}$ entering into expansion (19) of the determinant are Hermitian, provided that $q^{1/2}u_V$, w_V, and κ_V are real-valued for all V. Hence, any integrable model proposed for $N = 2$ has a physical interpretation.

Because all the operators under consideration become finite-dimensional, the fundamental problem of the theory of integrable systems, namely, the simultaneous diagonalization of the complete set of the commutative matrices $\mathbf{t}_{\alpha,\beta}$, or, at least, the evaluation of their spectra, makes sense. The simultaneous diagonalization can be performed in the case $N = 2$ because the matrices are Hermitian, while for $N > 2$, such a diagonalization is postulated.

3.2. Auxiliary Linear Problem in the Finite-Dimensional Case

We now consider the auxiliary problem in the finite-dimensional case:

$$\sum_S \langle\varphi_S|\mathbf{L}_{S|V} = 0. \tag{63}$$

System (63) is a system of equations in the N^{Δ}-dimensional vectors $\langle\varphi_S|$. Therefore, this is a system of ΔN^{Δ} $\mathbb{C}$-number equations, for which the solvability condition takes the form

$$\mathrm{Det}\,\mathbf{L} = 0, \tag{64}$$

where Det means the determinant of the $\Delta N^{\Delta}\times\Delta N^{\Delta}$-dimensional matrix $\mathbf{L}$. In this case,

$$\mathrm{Det}\,\mathbf{L} = \det\mathbf{j}, \tag{65}$$

where $\mathbf{j}$ is determinant (16), and $\det\mathbf{j}$ is evaluated for $\mathbf{j}$ taking as an $\Delta N^{\Delta}\times\Delta N^{\Delta}$ matrix. By virtue of (55), each $\langle\varphi_S|$ belongs to the zero subspace of the operator $\mathbf{j}$. Since the quantities u_V, w_V, κ_V are arbitrary parameters, condition (64) is a condition imposed on x and y.

To find a solution of auxiliary problem (63), we use the basis in which all $\mathbf{m}_{S,S_0}$ are diagonal matrices. In this case, Eq. (53) becomes trivial:

$$\langle\varphi_S| = \langle\varphi_{S_0}|\mathbf{m}_{S,S_0} = \langle\varphi_{S_0}|m_{S,S_0}, \tag{66}$$

where $m \in \mathbb{C}$ is an eigenvalue of the operator $\mathbf{m}$. Quasiperiodic conditions in terms of x and y are imposed on m_{S,S_0}. In this case, linear equation (14) takes the form

$$\begin{aligned}\langle\varphi|(m_{a,S_0} &+ m_{b,S_0}q^{1/2}\mathbf{u}_V + m_{c,S_0}\mathbf{w}_V\\ &+ m_{d,S_0}\kappa_V\mathbf{u}_V\mathbf{w}_V) = 0.\end{aligned} \tag{67}$$

Equation (67) has a trivial compatibility solution ($N\times N$ determinant for the Vth component)

$$\begin{aligned}m_{a,S_0}^N &- m_{b,S_0}^N\epsilon_N\mathbf{u}_V^N + m_{c,S_0}^N\epsilon_N\mathbf{w}_V^N\\ &+ m_{d,S_0}^N\kappa_V^N\mathbf{u}_V^N\mathbf{w}_V^N = 0,\end{aligned} \tag{68}$$

where

$$\epsilon_N \equiv \det\mathbf{x} \equiv \det\mathbf{z} = (-)^{N-1}. \tag{69}$$

The system of linear $\mathbb{C}$-number equations (68) coincides with quantum system (63), provided that the variables x and y in the periodic conditions imposed on φ_S are substituted by x^N and y^N in those imposed on m_{S,S_0}^N. System (68) can be written out in the matrix form

$$\sum_S m_{S,S_0}^N L_{S|V} = 0, \tag{70}$$

where the $\mathbb{C}$-number matrix elements $L_{S|V}$ are determined by the coefficients entering into Eq. (68).

Let

$$J(x^N, y^N) = \det\|L_{S|V}\|. \tag{71}$$

If $J(x^N, y^N) = 0$, but the rank of the matrix L is not less than $\Delta - 1$, then system (68) has a unique solution satisfying the normalization condition $m_{S_0, S_0} = 1$. Hence, the spectra of the operators are fixed, and the correct form of (66) is

$$\langle\varphi_S(\zeta)|\mathbf{m}_{S, S_0} = m_{S, S_0} q^{\zeta_S}\langle\varphi_{S_0}(\zeta)|. \tag{72}$$

Here, $m_{S_0, S_0} = 1$ and $\zeta_{S_0} = 0$. All the remaining m_{S, S_0} are fixed Nth roots of the solution m^N_{S, S_0} of Eq. (68), and all the remaining $\zeta_S \in \mathbb{Z}_N$. Therefore, the operators $\mathbf{m}$ are cyclic, i.e., $\mathbf{m}^N = m^N \in \mathbb{C}$. The matrix elements $\langle\varphi_S(\zeta)|\sigma'\rangle$ in basis (61) will be explicitly constructed below.

It is important to note that auxiliary linear problem (63) has a solution only if the quantities x^N and y^N satisfy the condition $J(x^N, y^N) = 0$. More specifically, equation $J(x^N, y^N) = 0$ defines an algebraic curve Γ_g with a point on it defined as $P \equiv (x^N, y^N)$. In what follows, we refer to the curve and the case $J(x^N, y^N) = 0$ as a classical spectral curve and the case of spectral parameters on the curve, respectively. On the contrary, arbitrary (x, y) will be referred to as free spectral parameters.

As follows from these arguments, the dimension of the linear space $\langle\varphi_S(\zeta)|$ defined by condition (55) and being the maximum possible dimension is equal to the number $N^{\Delta-1}$ of different sets ζ_S. Therefore, the condition $J(x^N, y^N) = 0$, being a necessary and sufficient solvability condition for system (63), yields the algebraic identity for free x and y:

$$\mathrm{Det}\|\mathbf{L}\| = \det \mathbf{j}(x, y) = J(x^N, y^N)^{N^{\Delta-1}}. \tag{73}$$

Equation (73) appears to have a simple combinatorial structure; indeed, it defines the method of evaluating the determinant $\mathrm{Det}\mathbf{L}$ of a matrix with the use of two different block decompositions of it.

The determinant $\det\mathbf{j}$ can in turn be evaluated differently. Indeed, because $\mathbf{j}$ contains the commutative set $\mathbf{t}_{\alpha, \beta}$ and one noncommutative pair $(\mathbf{U}_0, \mathbf{W}_0)$ [see, e.g., Eq. (41)],

$$\det \mathbf{j}(x, y) = \prod_{\mathbf{t}_{\alpha,\beta} = t_{\alpha,\beta}} \det_{\mathbf{U}_0, \mathbf{W}_0} \mathbf{j}(x, y) \tag{74}$$

in the basis of diagonal $\mathbf{t}$.

The $N \times N$ determinant of the noncommutative pair generates the Nth powers of x and y, while the product of the eigenvalues $\mathbf{t}_{\alpha, \beta} = t_{\alpha, \beta}$ contains $N^{\Delta-1}$ factors. Hence, because $J(x^N, y^N)$ is an irreducible polynomial in x^N and y^N being in general position, Eq. (73) is the identity that generates the Abelian algebra $\mathbf{t}_{\alpha, \beta}$ when expanded in terms of x^N and y^N:

$$\det_{\mathbf{U}_0, \mathbf{W}_0} \mathbf{j}(x, y) = J(x^N, y^N). \tag{75}$$

Equation (75) is a functional equation allowing us to evaluate the spectrum of $\mathbf{t}_{\alpha, \beta}$.

Rigorous proofs of the results presented in this section can be found in [5].

3.3. Examples

In this section, we consider some examples of the evaluation of $J(x^N, y^N)$ and the parametrization of $J(x^N, y^N) = 0$. For an arbitrary lattice with arbitrary u_V, w_V, and κ_V (so-called points in general position), the parametrization of u_V^N, w_V^N, κ_V^N, and $\mathbf{m}_S^N$ can be performed in general form within the framework of algebraic geometry. For a lattice with Δ vertices at points of intersection of Δ' lines, the equation $J(x^N, y^N) = 0$ defines a plane algebraic curve $\Gamma_g \ni P = (x^N, y^N)$ of the gth type, where $g = \Delta - \Delta' + 1$.[3] The parameters u_V^N, w_V^N, and κ_V^N and the quantities $\mathbf{m}_S^N$ are expressed in terms of the theta functions on $\mathrm{Jac}(\Gamma_g)$ and of primary forms on $\Gamma_g \times \Gamma_g$. For points in general position, Eq. (68) is equivalent to a combination of two Fay's identities. In addition to g modules and g components of an arbitrary vector $\vec{z} \in \mathrm{Jac}(\Gamma_g)$, divisors of meromorphic functions of x^N and y^N play an important part in this parametrization, serving in a sense as spectral parameters. The motion along $\vec{z}$, which changes u_V and w_V, represents an isospectral deformation of the theory because the coefficients in $J(x^N, y^N)$ remain unchanged. From the point of view of Poisson mechanics, this implies that the classical system (u_V^N, w_V^N), with the Poisson bracket $\{u_V^N, w_V^N\} = u_V^N w_V^N$ imposed on it, is an integrable system, where $\mathbf{m}_S^N$ is its Baker–Akhiezer function and $\vec{z}$ is its time. Some details concerning applications of algebraic geometry can be found in [3, 11–13].

The cases when the initial points u_V, w_V, and κ_V are not in general position (i.e., when the curve Γ_g degenerates) are of special interest. Because of this, we will not consider curves of a fairly high kind here. The analysis of the classical system within the framework of algebraic geometry can be found in [3, 11].

3.3.1. Trivial example. Let us consider a square lattice with $N_2 = N_3 = 1$ and $\Delta = 1$. According to Eqs. (35) and (36),

$$\mathbf{L} \equiv \mathbf{j}(x, y) = 1 + yq^{1/2}\mathbf{u} + x\mathbf{w} + xy\kappa\mathbf{u}\mathbf{w}, \tag{76}$$

[3] The "classical" curve Γ should not be confused with Γ^Q, which can be referred to as a quantum curve. Points of these curves are given by pairs (x^N, y^N) and (x, y), respectively.

where $\mathbf{u}$ and $\mathbf{w}$ is the single noncommutative pair. Such a determinant was evaluated above [see Eqs. (67) and (68)]:

$$\begin{aligned}\operatorname{Det}\mathbf{L} \equiv \det_{\mathbf{u},\mathbf{w}}\mathbf{j} &= 1 - y^N\epsilon_N\mathbf{u}^N + x^N\epsilon_N\mathbf{w}^N \\ &+ x^N y^N\kappa^N\mathbf{u}^N\mathbf{w}^M \equiv J(x^N, y^N).\end{aligned} \tag{77}$$

Equation $J(x^N, y^N) = 0$ defines a curve of the zero type on (x^N, y^N).[4]

3.3.2. Uniform square lattice. For a square $N_2 \times N_3$ lattice with uniform u_V, w_V, and κ_V, we can set, without loss of generality,

$$u_{n_2, n_3} = -q^{-1/2}, \quad w_{n_2, n_3} = -1, \quad \kappa_{n_2, n_3} = \kappa. \tag{78}$$

In notations (35), Eqs.(68) for $\mathbf{m}_S^N$ take the form (the sign of S_0 is omitted)

$$\mathbf{m}^N_{n_2, n_3} - \mathbf{m}^N_{n_2, n_3+1} - \mathbf{m}^N_{n_2-1, n_3} - \mathbf{m}_{n_2-1, n_3+1}\kappa^N = 0, \tag{79}$$

with the boundary conditions $\mathbf{m}^N_{-1, n_3} = x^N\mathbf{m}^N_{N_2-1, n_3}$ and $\mathbf{m}^N_{n_2, N_3} = y^N\mathbf{m}^N_{n_2, 0}$. Because Eq. (79) is the first-order recursion with uniform coefficients, its solution can be sought in the form $\mathbf{m}_{n_2, n_3} = \lambda^{-n_2}\mu^{n_3}$, where

$$1 - \lambda - \mu - \kappa^N\lambda\mu = 0, \quad \lambda^{N_2} = x^N, \quad \mu^{N_3} = y^N. \tag{80}$$

Therefore, if x and y are points in general position, then

$$J(x^N, y^N) = \prod_{\lambda^{N_2} = x^N,\, \mu^{N_3} = y^N} (1 - \lambda - \mu - \kappa^N\lambda\mu). \tag{81}$$

As an example, we now write the functional equation for the spectrum of $\mathbf{j}$ (38) in the case when $N = 2$. According to normalization condition (78), Eqs. (39) reduce to

$$\mathbf{U}_{n_2} = \prod_{n_3=0}^{N_3-1}\mathbf{x}_{n_2, n_3}, \quad \mathbf{W}_{n_3} = \prod_{n_2=0}^{N_2-1}\mathbf{z}_{n_2, n_3}, \tag{82}$$

so that $\mathbf{U}_0^2 = \mathbf{W}_0^2 = 1$ and

$$\begin{aligned}\mathbf{j}(x, y) &= \sum_{\nu_1, \nu_2}(x\mathbf{W}_0)^{\nu_2}(y\mathbf{U}_0)^{\nu_1}\mathbf{t}_{\nu_1, \nu_2} \\ &\equiv t_{0,0}(x^2, y^2) - x\mathbf{W}_0 t_{0,1}(x^2, y^2) - y\mathbf{U}_0 t_{1,0}(x^2, y^2) \\ &- ixy\mathbf{W}_0\mathbf{U}_0 t_{1,1}(x^2, y^2).\end{aligned} \tag{83}$$

As was noted above, for $N = 2$ and a real-valued κ, all matrix elements $\mathbf{j}_{\nu_2, \nu_3}$ and, therefore, polynomials $t_{j,k}(x^2, y^2)$ are Hermitian. The determinant of $\mathbf{j}$ on the subspace of the operators $\mathbf{U}_0$ and $\mathbf{W}_0$ can be expressed in terms of $t_{j,k}(x^2, y^2)$:

$$\begin{aligned}\det_{\mathbf{U}_0, \mathbf{W}_0}\mathbf{j}(x, y) &\equiv t^2_{0,0}(x^2, y^2) - x^2 t^2_{0,1}(x^2, y^2) \\ &- y^2 t^2_{1,0}(x^2, y^2) - x^2y^2 t^2_{1,1}(x^2, y^2).\end{aligned} \tag{84}$$

Equating expression (84) to $J(x^2, y^2)$ given by Eq. (81) for $N = 2$, we arrive at a functional equation. One should pay special attention to the polynomial structure of $t_{j,k}(x^2, y^2)$, which follows from expansions (38) and (40). It is convenient to fix the basis $(\mathbf{U}_0, \mathbf{W}_0)$ and set, for example, $\mathbf{U}_0 = \sigma_2$ and $\mathbf{W}_0 = \sigma_1$. In this case,

$$\begin{aligned}\mathbf{j}(x, y) &= \begin{pmatrix} t_{0,0} + xyt_{1,1} & xt_{0,1} + iyt_{1,0} \\ xt_{0,1} - iyt_{1,0} & t_{0,0} - xyt_{1,1}\end{pmatrix} \\ &\equiv \begin{pmatrix} A(x, y) & B(x, y) \\ C(x, y) & D(x, y)\end{pmatrix},\end{aligned} \tag{85}$$

so that $\det_{\mathbf{U}_0, \mathbf{W}_0}\mathbf{j} = AD - BC$.

As an example of a particular solution of the functional equation, we consider a lattice having the even dimension $N_3 = 2M$ in one direction. Let

$$f(x^2, y) = \prod_{\lambda^{N_2} = x^2,\, \mu^M = y}(1 - \lambda - \mu - \kappa^2\lambda\mu). \tag{86}$$

It is evident that $J(x^2, y^2) = f(x^2, y)f(x^2, -y)$, and the functional equation has the solution

$$t_{0,0} + xyt_{1,1} = f(x^2, -y), \quad t_{1,0} = t_{0,1} = 0. \tag{87}$$

We find only the simplest solution of the functional equation, which has $2^{N_2N_3-1}$ different solutions. In other words, for an arbitrary solution with the proper polynomial structure of $t_{j,k}(x^2, y^2)$, there exists a unique spin-lattice state with appropriate eigenvalues $\mathbf{t}_{\nu_2, \nu_3}$.

3.3.3. Nonuniform square lattice with a rational parametrization. We now consider a specific auxiliary nonuniform square lattice. Instead of evaluating $J(x^N, y^N)$ at points (x, y) in general position, we propose here a method of parametrizing linear equations (68) in coordinates (35):

$$\begin{aligned}\mathbf{m}^N_{n_2, n_3} - \mathbf{m}^N_{n_2, n_3+1}\epsilon_N\mathbf{u}^N_{n_2, n_3} + \mathbf{m}^N_{n_2-1, n_3}\epsilon_N\mathbf{w}^N_{n_2, n_3} \\ + \mathbf{m}^N_{n_2-1, n_3+1}\kappa^N_{n_2, n_3}\mathbf{u}^N_{n_2, n_3}\mathbf{w}^N_{n_2, n_3} = 0.\end{aligned} \tag{88}$$

The nonuniformity of the lattice is described by the following redefinition of the lattice parameters in terms of

[4] As a remark to Footnote 3, it is worth noting that the equality $J(x^N, y^N) = 0$ considered as an equation in (x, y) defines a special Baxter curve γ_G of the Gth type with the Baxter module $k^2 = -\kappa^N$, where $G = (N-1)^2$.

$(2N_2 + 2N_3)$ $\mathbb{C}$ numbers $(Y_{n_2}, Y'_{n_2}, Z_{n_2}, Z'_{n_2})$ and $(N_2 + N_3)$ auxiliary scale factors $(\xi_{2,n_2}, \xi_{3,n_3})$:

$$\begin{aligned}\mathbf{u}^N_{n_2,n_3} &= \epsilon_N \xi^N_{3,n_3} \frac{Y'_{n_2} - Z'_{n_3}}{Y'_{n_2} - Z_{n_3}},\\ \mathbf{w}^N_{n_2,n_3} &= -\epsilon_N \xi^N_{2,n_2} \frac{Y_{n_2} - Z_{n_3}}{Y'_{n_2} - Z_{n_3}},\\ \kappa^N_{n_2,n_3} &= -\frac{(Y_{n_2} - Z'_{n_3})(Y'_{n_2} - Z_{n_3})}{(Y_{n_2} - Z_{n_3})(Y'_{n_2} - Z'_{n_3})}.\end{aligned} \tag{89}$$

It is easy to prove that, for such values of the parameters, the general unnormalized solution of system (88) is

$$\mathbf{m}^N_{n_2-1,n_3} = \mathbf{m}^N_{n_2-1,n_3}(X) = \prod_{j_2=0}^{n_2-1} \xi^N_{2,j_2} \frac{X - Y_{j_2}}{X - Y'_{j_2}} \prod_{j_3=0}^{n_3-1} \frac{1}{\xi^N_{3,j_3}} \frac{X - Z_{j_3}}{X - Z'_{j_3}}, \tag{90}$$

where X is a free parameter determining the quantities x^N and y^N on the curve $J(x^N, y^N) = 0$ under the closure of Eq. (90):

$$x^{-N} = \prod_{n_2 \in \mathbb{Z}_{N_2}} \xi^N_{2,n_2} \frac{X - Y_{n_2}}{X - Y'_{n_2}}, \tag{91}$$

$$y^N = \prod_{n_3 \in \mathbb{Z}_{N_3}} \frac{1}{\xi^N_{3,n_3}} \frac{X - Z_{n_3}}{X - Z'_{n_3}}.$$

Parametrization (89) and (90) holds, for example, under the reduction of the classical curve Γ_g, with $g = (N_2 - 1)(N_3 - 1)$, into a sphere with distinguished points $(Y_{n_2}, \ldots, Z'_{n_3})$. In this case, the theta functions reduce to unities, while the primary form of the two divisors X and Y reduces to the primary form on the sphere, $(X - Y)/\sqrt{dXdY}$. Such reductions are described in detail in [12]. Parametrization (89) is not unique. As is shown below, it represents, in a sense, the "zero-soliton" parametrization.

3.3.4. Checkerboard lattice. Finally, we consider the case of the square lattice when Γ_g is factored into a torus. Such a factorization is natural for a periodic lattice of parameters with the step $M = 2$: $\kappa_{n_2,n_3} = \kappa_{n_2 \bmod 2, n_3 \bmod 2}$, $u_{n_2,n_3} = u_{n_2 \bmod 2, n_3 \bmod 2}$, and $w_{n_2,n_3} = w_{n_2 \bmod 2, n_3 \bmod 2}$, where N_2 and N_3 are even. In order to evaluate $J(x^N, y^N)$, we find its Fourier transform [in much the same way as for (81)], make the substitution $\mathbf{m}^N_{n_2-2,n_3} = \lambda \mathbf{m}^N_{n_2,n_3}$ and $\mathbf{m}^N_{n_2,n_3+2} = \mu \mathbf{m}^N_{n_2,n_3}$, and evaluate the determinant

$$\chi_4(\lambda,\mu) = \det\begin{vmatrix} 1 & -\mu\epsilon_N \mathbf{u}^N_{0,1} & \epsilon_N \mathbf{w}^N_{1,0} & \mu\kappa^N_{1,1}\mathbf{u}^N_{1,1}\mathbf{w}^N_{1,1} \\ -\epsilon_N \mathbf{u}^N_{0,0} & 1 & \kappa^N_{1,0}\mathbf{u}^N_{1,0}\mathbf{w}^N_{1,0} & \epsilon_N \mathbf{w}^N_{1,1} \\ \lambda\epsilon_N \mathbf{w}^N_{0,0} & \lambda\mu\kappa^N_{0,1}\mathbf{u}^N_{0,1}\mathbf{w}^N_{0,1} & 1 & -\mu\epsilon_N \mathbf{u}^N_{1,1} \\ \lambda\kappa^N_{0,0}\mathbf{u}^N_{0,0}\mathbf{w}^N_{0,0} & \lambda\epsilon_N \mathbf{w}^N_{0,1} & -\epsilon_N \mathbf{u}^N_{1,0} & 1 \end{vmatrix}. \tag{92}$$

If the point (x^N, y^N) is in general position, the result is

$$J(x^N, y^N) = \prod_{\lambda^{N_2/2} = x^N,\ \mu^{N_3/2} = y^N} \chi_4(\lambda,\mu). \tag{93}$$

The parameter lattice can be simplified by using the checkerboard structure:

$$\begin{aligned}\epsilon_N \mathbf{u}^N_{0,0} &= \epsilon_N \mathbf{u}^N_{1,1} = b, \quad \epsilon_N \mathbf{w}^N_{0,0} = \epsilon_N \mathbf{w}^N_{1,1} = -a,\\ \kappa^N_{0,0}\mathbf{u}^N_{0,0}\mathbf{w}^N_{0,0} &= \kappa^N_{1,1}\mathbf{u}^N_{1,1}\mathbf{w}^N_{1,1} = -c,\\ \epsilon_N \mathbf{u}^N_{1,0} &= \epsilon_N \mathbf{u}^N_{0,1} = b', \quad \epsilon_N \mathbf{w}^N_{1,0} = \epsilon_N \mathbf{w}^N_{0,1} = -a',\\ \kappa^N_{1,0}\mathbf{u}^N_{1,0}\mathbf{w}^N_{1,0} &= \kappa^N_{0,1}\mathbf{u}^N_{0,1}\mathbf{w}^N_{0,1} = -c'.\end{aligned} \tag{94}$$

For $N = 2$, the physical regime corresponds to real-valued parameters a, a', b, b', c, and c'. After performing the change of variables

$$\begin{aligned} aa'\lambda = \frac{u^2}{w^2}, \quad bb'\mu = \frac{v^2}{w^2}, \quad cc'\lambda\mu = u^2 v^2,\\ \left(4h\frac{uv}{w}\right)^2 = (c + c' + ab' + a'b)^2 \lambda\mu,\end{aligned} \tag{95}$$

$$h = \sqrt{k^{-2} + \left(\frac{w - w^{-1}}{2}\right)^2}, \quad k^{-2} = h^2 - \left(\frac{w - w^{-1}}{2}\right)^2, \tag{96}$$

we arrive at the equation

$$\chi_4(\lambda\mu) = \chi_2(u, v, w, k)\chi_2(-u, v, w, k), \tag{97}$$

where

$$\chi_2(u, v, w, k) = 1 - \frac{u^2}{w^2} - \frac{v^2}{w^2} + u^2 v^2 - 4h\frac{uv}{w}. \tag{98}$$

3.4. Thermodynamic Limit

The spectrum of the integrals of the commutative set entering into the functional $\mathbf{j}(x, y)$ is determined by Eq. (75). It seems impossible to explicitly solve this equation for a large auxiliary lattice; however, for some states we can make certain asymptotic estimates as N_2 and N_3 tend to infinity. This limit will be referred to as a thermodynamic limit, even though the system under consideration is not a statistical one. Details of the calculations given below in this section can be found in [9].

3.4.1. Uniform lattice. First, we consider the thermodynamic limit for a uniform square lattice with $N = 2$, when $J(x^N, y^N)$ is given by formula (81). Let the limiting values

$$\lambda = \lim_{N_2 \mapsto \infty} x^{N/N_2} \quad \text{and} \quad \mu = \lim_{N_3 \mapsto \infty} y^{N/N_3} \tag{99}$$

be real numbers [see Eq. (80)]. Introducing

$$\chi_1(\lambda, \mu, \kappa^N) = 1 - \lambda - \mu - \kappa^N \lambda\mu, \tag{100}$$

we can write Eq. (81) in the form

$$\begin{aligned} J &= J(\lambda, \mu) \\ &= \prod_{n_2 = 0}^{N_2 - 1} \prod_{n_3 = 0}^{N_3 - 1} \chi_1(\lambda e^{2\pi i n_2/N_2}, \mu e^{2\pi i n_3/N_3}, \kappa^N). \end{aligned} \tag{101}$$

In particular, for J on the "curve" $J = 0$, this implies that there exist distinguished pairs (one or two) $(n_2, n_3) = (m_2, m_3)$ such that $\chi_1(\lambda e^{2\pi i m_2/N_2}, \mu e^{2\pi i m_3/N_3}) = 0$. In this case, we can define the nonzero quantity

$$\begin{aligned} &J'(\lambda, \mu) \\ &= \prod_{(n_2, n_3) \neq (m_2, m_3)} \chi_1(\lambda e^{2\pi i n_2/N_2}, \mu e^{2\pi i n_3/N_3}, \kappa^N). \end{aligned} \tag{102}$$

If $J(x^N, y^N)$ is real-valued, i.e., $\overline{J(x^N, y^N)} = J(x^N, y^N)$, then

$$J \quad \text{or} \quad J' \sim e^{N_2 N_3 \mathfrak{f}} \tag{103}$$

as N_2 and N_3 tend to infinity. Here,

$$\begin{aligned} \mathfrak{f}(\lambda, \mu, \kappa^N) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi_2 \\ &\times \int_0^{2\pi} d\phi_3 \log\left|1 - \lambda e^{i\phi_2} - \mu e^{i\phi_3} - \kappa^N \lambda\mu e^{i\phi_2 + i\phi_3}\right|. \end{aligned} \tag{104}$$

The quantity J approaches J' asymptotically because logarithmic singularities are integrable.

Substituting asymptotic expression (103) into functional equation (75), we come to the conclusion that there exist states for Eq. (85) ($N = 2$) such that the maximum eigenvalues of the operators A, B, C, and D satisfy the asymptotic condition

$$\max(A, B, C, D) \sim e^{N_2 N_3 \mathfrak{f}/2}. \tag{105}$$

The states are, of course, different for A, B, C, and D. The evaluation of expression (104) depends on whether J is considered on the curve or outside of it. In these cases, the argument of the logarithmic function has two or no zeros on the torus, respectively.

In the first case, we take

$$\lambda = \frac{\sin\beta_2}{\sin\beta_1}, \quad \mu = \frac{\sin\beta_3}{\sin\beta_1}, \quad \kappa^N = \frac{\sin\beta_0 \sin\beta_1}{\sin\beta_2 \sin\beta_3}, \tag{106}$$

where, by definition,

$$\beta_0 = \pi - \beta_1 - \beta_2 - \beta_3. \tag{107}$$

The zeros of χ_1 are determined from the equation

$$\begin{aligned} &\chi_1\left(\frac{1 - e^{-2i\beta_2}}{1 - e^{2i\beta_1}}, \frac{1 - e^{2i\beta_3}}{1 - e^{-2i\beta_1}}\right) \\ &= \chi_1\left(\frac{1 - e^{2i\beta_2}}{1 - e^{-2i\beta_1}}, \frac{1 - e^{-2i\beta_3}}{1 - e^{2i\beta_1}}\right) = 0. \end{aligned} \tag{108}$$

Because of the evident symmetry

$$\begin{aligned} \mathfrak{f}(\beta_1, \beta_2, \beta_3) &= \mathfrak{f}(\pi + \beta_1, \beta_2, \beta_3) \\ = \mathfrak{f}(\beta_1, \pi + \beta_2, \beta_3) &= \mathfrak{f}(\beta_1, \beta_2, \pi + \beta_3) \\ &= \mathfrak{f}(-\beta_1, -\beta_2, -\beta_3), \end{aligned} \tag{109}$$

any set β_1, β_2, and β_3 can be reduced to the canonical form:

$$\begin{gathered} 0 \le \beta_1 + \beta_2 < \pi, \quad 0 \le \beta_1 + \beta_3 < \pi, \\ 0 \le \beta_2 + \beta_3 < \pi. \end{gathered} \tag{110}$$

In this case,

$$\begin{aligned} \mathfrak{f} &= \Im(\beta_0) + \Im(\beta_1) + \Im(\beta_2) \\ &+ \Im(\beta_3) - \log|2\sin\beta_1|, \end{aligned} \tag{111}$$

where the function $\Im$ is related to the dilogarithm:

$$\begin{gathered} \Im(\beta) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{\beta} \alpha \cot\alpha \, d\alpha = \frac{\beta}{\pi}\log|2\sin\beta| + \sum_{m=1}^{\infty} \frac{\sin 2m\beta}{2\pi m^2}, \\ -\pi < \beta < \pi. \end{gathered} \tag{112}$$

To obtain $\mathfrak{f}$ for a set β_1, β_2, and β_3 different from (110), one can use the simple symmetry properties of the function $\Im$:

$$\begin{aligned} \Im(\beta) + \Im(-\beta) &= 0, \\ \Im(\beta) + \Im(\pi - \beta) &= \log|2\sin\beta|. \end{aligned} \tag{113}$$

Values of κ^N can be within an unphysical range ($\kappa^N < 0$) as well as within a physical range ($\kappa > 0$).

If the equation $\chi_1(ae^{i\phi_2}, be^{i\phi_3}) = 0$ has no solutions (or only one solution, for example, $\chi_1(a, b) = 0$), we can introduce the parametrization

$$\begin{gathered}\lambda = \frac{\sinh\beta_2}{\sinh\beta_1}, \quad \mu = \frac{\sinh\beta_3}{\sinh\beta_1}, \\ \kappa^N\lambda\mu = \frac{\sinh(\beta_1+\beta_2+\beta_3)}{\sinh\beta_1},\end{gathered} \tag{114}$$

where $\beta_j \in \mathbb{R} \bmod i\pi$. In this case,

$$\mathfrak{f} = \log\max(1, |\lambda|, |\mu|, |\kappa^N\lambda\mu|). \tag{115}$$

It is noteworthy that expression (111) is similar to that for the statistical sum [14] in the Zamolodchikov–Bazhanov–Baxter model [15, 16]. This similarity is not accidental. Expression (111) has no singularities, which could indicate a phase transition. This is consistent with the fact that the Zamolodchikov–Bazhanov–Baxter model is a critical model [17]. The absence of singularities is due to the simple form of integral (104). An elementary method of complicating the integral will be described in the next section.

3.4.2. Checkerboard lattice. We now consider a checkerboard lattice, with $J(x^N, y^N)$ given by Eqs. (93), (97), and (98). In this case, expression (103) and assumption (105) hold, but the function $\mathfrak{f}$ takes the form

$$\mathfrak{f} = \frac{1}{2(2\pi)^2}\int\int_0^{2\pi} d\phi d\phi' \log\left|\chi_2(ue^{i\phi}, ve^{i\phi'})\right|. \tag{116}$$

Because the function $\mathfrak{f}$ depends on $|u|$ and $|v|$, we can take all u, v, w, and h as real positive numbers. It is easy to find some symmetry properties of $\mathfrak{f}$. For example, using $\chi_2(u, v, w, k) = \chi_2(v, u, w, k)$ or $\chi_2(u, v, w, k) = -\frac{u^2}{w^2}\chi_2(u^{-1}, v, w^{-1}, k)$, we have

$$\mathfrak{f}(u, v, w, k) = \frac{1}{2}\log\frac{uv}{w} + \mathfrak{f}_0(u, v, w, k), \tag{117}$$

where

$$\begin{aligned}\mathfrak{f}_0(u, v, w, k) &= \mathfrak{f}_0(v, u, w, k) \\ = \mathfrak{f}(u^{-1}, v, w^{-1}, k) &= \mathfrak{f}(u^{-1}, v^{-1}, w, k).\end{aligned} \tag{118}$$

We integrate (116) for $u = v = 1$, when the following three regimes are possible.

In the first place, if $0 < k \le 1$ (i.e., $h \ge \frac{w+w^{-1}}{2}$), then

$$\mathfrak{f}_0(w, k) = \mathfrak{T}(w, k), \tag{119}$$

where

$$\begin{gathered}\mathfrak{T}(w, k) \overset{\text{def}}{=} \frac{1}{\pi}\int_0^{\pi/2} d\phi \\ \times\log\left(2\sqrt{k^{-2} + \left(\frac{w-w^{-1}}{2}\right)^2} + 2\sqrt{k^{-2} - \sin^2\phi}\right).\end{gathered} \tag{120}$$

In particular, for $k = 1$ and $w = \tan\beta_1$, with $0 \le \beta_1 \le \pi/2$, we have

$$\begin{aligned}\mathfrak{T}(w = \tan\beta_1, k = 1) &= \frac{2}{\pi}\text{Catalan} + \frac{1}{\pi}\int_0^{\pi/2-2\beta_1}\frac{\alpha d\alpha}{\cos\alpha} \\ &= 2\Im(\beta_1) + 2\Im(\pi/2-\beta_1) - \frac{1}{2}\log(2\sin 2\beta_1).\end{aligned} \tag{121}$$

Here, $\Im$ is defined by Eq. (112), and Catalan $\overset{\text{def}}{=} \sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)^2} \sim 0.9159655942$. In this case, $\mathfrak{f}$ is given by Eq. (11) with $\beta_0 = \beta_1$ and $\beta_2 = \beta_2 = \pi/2 - \beta_1$. Thus, expression (120) is in a sense an elliptic extension of the dilogarithm. The derivative of Eq. (119) [see the relations between h, k, and w in Eq. (96)]

$$\begin{gathered}\frac{\partial\mathfrak{f}_0(w, k)}{\partial k^{-1}} = \frac{h^{-1}}{\pi}K(k), \\ \text{where} \quad K(k) \overset{\text{def}}{=} \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\end{gathered} \tag{122}$$

has a logarithmic singularity at $k = 1$. This looks like a phase transition in lattice integrable models.

The second regime corresponds to $0 \le k^{-1} \le 1$ (i.e., $\frac{|w-w^{-1}|}{2} \le h \le \frac{w+w^{-1}}{2}$). In this case,

$$\begin{gathered}\mathfrak{f}_0(w, k) = \frac{1}{2}|\log w| \\ + \frac{1}{\pi}\int_0^{\arcsin(k^{-1}),} d\phi\log\left(\frac{h + \sqrt{k^{-2}-\sin^2\phi}}{\sqrt{\left(\frac{w-w^{-1}}{2}\right)^2 + \sin^2\phi}}\right)\end{gathered} \tag{123}$$

where $0 \le \arcsin(k^{-1}) \le \pi/2$ and

$$\frac{\partial\mathfrak{f}_0(w, k)}{\partial k^{-1}} = \frac{k^{-1}h^{-1}}{\pi}K(k^{-1}). \tag{124}$$

In the third regime, $-\left(\frac{w-w^{-1}}{2}\right)^2 \le -k^{-2} \le 0$ (i.e., $0 \le h \le \frac{|w-w^{-1}|}{2}$) and

$$\mathfrak{f}(w, k) = \frac{1}{2}|\log w|. \qquad (125)$$

For the more complicated case of $u, v \neq 1$, we failed to find a reasonable expression holding in an entire four-dimensional space (u, v, w, k) (especially in the case of the function χ_2 having four zeros in the integration domain). Such solutions for various particular cases can be found in [9]. It is worth noting that there are some indications of phase transitions in the case of $u, v \neq 1$, namely, singularities similar to that of the complete elliptic integral $K(k)$ at $k = 1$.

4. ZERO-CURVATURE REPRESENTATION

In the previous sections, we constructed three-dimensional analogues of chains and of auxiliary transfer matrices. We now consider three-dimensional analogues of quantum intertwining operators, namely, analogues of Eq. (5) and of the quantities defined by Eqs. (6)–(9).

4.1. Operator $\mathfrak{R}$

4.1.1. Formulation of the zero-curvature representation. Let us consider a fragment of the auxiliary plane, including a triangle shown on the left side of Fig. 7. The vertices of the triangle are numbered by 1, 2, and 3, while the sites are indexed with b, c, d, e, f, g, and h. According to the rules of Fig. 1, three linear relations of the form $l_V = \sum_S \varphi_S \mathbf{L}_{S|V}$ are associated with the triangle. Here,

$$S \in (b, c, d, e, f, g, h), \quad V \in (1, 2, 3), \qquad (126)$$

and, for the indexing defined above,

$$\|\mathbf{L}_{S|V}\| = \begin{pmatrix} 0 & \mathbf{w}_2 & \kappa_3\mathbf{u}_3\mathbf{w}_3 \\ 1 & 0 & 1 \\ \kappa_1\mathbf{u}_1\mathbf{w}_1 & q^{1/2}\mathbf{u}_2 & 0 \\ q^{1/2}\mathbf{u}_1 & 0 & 0 \\ 0 & \kappa_2\mathbf{u}_2\mathbf{w}_2 & 0 \\ 0 & 0 & \mathbf{w}_3 \\ \mathbf{w}_0 & 1 & q^{1/2}\mathbf{u}_3 \end{pmatrix}. \qquad (127)$$

An alternative variant for the crossing lines is shown on the right side of Fig. 7. As before, the vertices are numbered by 1, 2, and 3. However, to distinguish the sides of Fig. 7, we associate pairs $\mathbf{u}'_V$ and $\mathbf{w}'_V$ with the right vertices. The sites are now indexed with a, b, c, d, e, f, and g. The outer sites on the sides are the same, while the inner

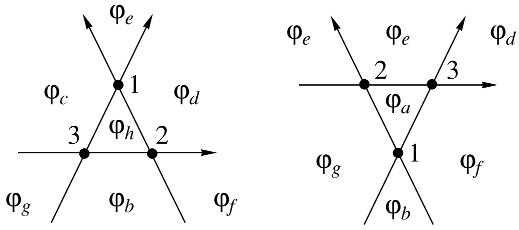

Fig. 7. Zero curvature representation as equivalence of two triangles.

site h is replaced by a. The triple of linear relations for the right side has the form $l'_V = \sum_{S'} \varphi_{S'} \mathbf{L}'_{S'|V}$, where

$$S' \in (b, c, d, e, f, g, a), \quad V \in (1, 2, 3); \qquad (128)$$

$$\|\mathbf{L}'_{S'|V}\| = \begin{pmatrix} \mathbf{w}'_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{1/2}\mathbf{u}'_3 \\ 0 & q^{1/2}\mathbf{u}'_2 & 1 \\ \kappa_1\mathbf{u}'_1\mathbf{w}'_1 & 0 & \kappa_3\mathbf{u}'_3\mathbf{w}'_3 \\ 1 & \mathbf{w}'_2 & 0 \\ q^{1/2}\mathbf{u}'_1 & \kappa_2\mathbf{u}'_2\mathbf{w}'_2 & \mathbf{w}'_3 \end{pmatrix}. \qquad (129)$$

It is important that the parameters κ_V in the matrices $\mathbf{L}$ and $\mathbf{L}'$ are the same.

Let $x, y \in (b, c, d, e, f, g)$ and $x \neq y$. The number of such nonequivalent pairs is equal to 15. Let $M_{x,y,h}$ ($M'_{x,y,a}$) be the minor formed by the rows x, y, and h (x, y, and a) of the matrix $\mathbf{L}$ ($\mathbf{L}'$).

Definition 3. *The relations*

$$M_{x,y,h}\mathbf{u}_1^{-1} = M'_{x,y,a}\mathbf{u}_1'^{-1} \quad \forall x, y \qquad (130)$$

will be referred to as a zero-curvature representation.

The multipliers of the minors are normalizing factors for $x = c$ and $y = e$. In this case, Eq. (130) reduces to $q^{1/2} = q^{1/2}$ and we arrive at a system of fourteen equations.

The zero-curvature representation defined above looks fairly like the so-called local Yang–Baxter equation [18], because two Yang–Baxter triangles are compared. In fact, the zero-curvature representation defined is close to the equivalence condition for electrical connections [19]. A classification of solutions of the local Yang–Baxter equation can be found in [20].

4.1.2. Solution. Relations (130) should be considered as a system of fourteen equations in the six unknowns $\mathbf{u}'_1$, $\mathbf{w}'_1$, $\mathbf{u}'_2$, $\mathbf{w}'_2$, $\mathbf{u}'_3$, and $\mathbf{w}'_3$. In spite of the overdetermination, this system has the unique solution

$$\begin{aligned} &\mathbf{w}'_1 = \mathbf{w}_2\Lambda_3, \quad \mathbf{w}'_2 = \Lambda_3^{-1}\mathbf{w}_1, \quad \mathbf{w}'_3 = \Lambda_2^{-1}\mathbf{u}_1^{-1}, \\ &\mathbf{u}'_1 = \Lambda_2^{-1}\mathbf{w}_3^{-1}, \quad \mathbf{u}'_2 = \Lambda_1^{-1}\mathbf{u}_3, \quad \mathbf{u}'_3 = \mathbf{u}_2\Lambda_1, \end{aligned} \qquad (131)$$

where

$$\Lambda_1 = \mathbf{u}_1^{-1}\mathbf{u}_3 - q^{1/2}\mathbf{u}_1^{-1}\mathbf{w}_1 + \kappa_1\mathbf{w}_1\mathbf{u}_2^{-1},$$

$$\Lambda_2 = \frac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_3^{-1} + \frac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1}\mathbf{w}_2^{-1} - q^{-1/2}\frac{\kappa_1\kappa_3}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_2^{-1}, \quad (132)$$

$$\Lambda_3 = \mathbf{w}_1\mathbf{w}_3^{-1} - q^{1/2}\mathbf{u}_3\mathbf{w}_3^{-1} + \kappa_3\mathbf{w}_2^{-1}\mathbf{u}_3.$$

Moreover, the quantities $\mathbf{u}'_1, \ldots, \mathbf{w}'_1$ given by Eqs. (131) and (132) form the same local Weyl algebra (13) as the original $\mathbf{u}_1, \ldots, \mathbf{w}_3$ do, i.e., an automorphism of the local Weyl algebra takes place.

Definition 4. *We define the operator* $\mathfrak{R}_{1,2,3}$ *implementing the similarity transformation, i.e., the automorphism* $(\mathbf{u}_V, \mathbf{w}_V) \mapsto (\mathbf{u}'_V, \mathbf{w}'_V)$ *given by Eqs. (131) and (132)*:

$$\mathfrak{R}_{1,2,3}\mathbf{u}_V = \mathbf{u}'_V\mathfrak{R}_{1,2,3}, \quad \mathfrak{R}_{1,2,3}\mathbf{w}_V = \mathbf{w}'_V\mathfrak{R}_{1,2,3}, \quad V = 1, 2, 3. \quad (133)$$

In fact, the operator $\mathfrak{R}$ is defined as a mapping on the ring of rational functions of the local Weyl algebra. Definition (133) in terms of the adjoint action of an operator is fairly relative until the Hilbert space is specified. In what follows, we denote such mappings by gothic symbols.

Statement 3. *Automorphism defined by Eq. (133) satisfies the tetrahedron equation*

$$\mathfrak{R}_{1,2,3}\mathfrak{R}_{1,4,5}\mathfrak{R}_{2,4,6}\mathfrak{R}_{3,5,6} = \mathfrak{R}_{3,5,6}\mathfrak{R}_{2,4,6}\mathfrak{R}_{1,4,5}\mathfrak{R}_{1,2,3}. \quad (134)$$

Proof. Let us consider four nonparallel lines such that the triangles formed by any three of the lines be oriented as the triangle on the lift side of Fig. 7. The vertices of this figure are numbered from 1 to 6 so that the triples (1, 2, 3), (1, 4, 5), (2, 4, 6), and (3, 5, 6) correspond to the vertices of the partial triangles. Displacing the lines of this graph, we can obtain another graph, in which all the partial triangles are oriented as the triangle on the right side of Fig. 7. There exist two sets of such displacements evidently corresponding to the transformations $\mathfrak{T}_{\text{left}} = \mathfrak{R}_{1,2,3}\mathfrak{R}_{1,4,5}\mathfrak{R}_{2,4,6}\mathfrak{R}_{3,5,6}$ and $\mathfrak{T}_{\text{right}} = \mathfrak{R}_{3,5,6}\mathfrak{R}_{2,4,6}\mathfrak{R}_{1,4,5}\mathfrak{R}_{1,2,3}$. However, we can evaluate the result of this transformation, regardless of the order of the displacements, using only the initial and final graphs. In this case, we come to the equivalence problem for 6×6 minors of the two 11×6 matrices corresponding to the linear problems, similar to system (130), for the two configurations. Because this overdetermined system has the unique solution $\mathfrak{T}$, it is evident that $\mathfrak{T} = \mathfrak{T}_{\text{left}} = \mathfrak{T}_{\text{right}}$.

Therefore, we obtain a three-dimensional intertwiner that satisfies the three-dimensional analogue of the Yang–Baxter equation.

4.1.3. $\mathfrak{R}$ as an evolution operator. The operator $\mathfrak{R}$ can be treated as the simplest evolution operator similar to two-dimensional operator (9) in the case when $m = 2$. We now consider an auxiliary lattice formed by three lines on the torus. Such a lattice was described above (see Fig. 3). Because of the cyclic boundary conditions, the triangle shown in Fig. 3 is equivalent to both the triangles in Fig. 7. Therefore, the displacement associated with the transformation $\mathfrak{R}_{1,2,3}$ does not affect such a lattice. It follows from zero-curvature condition (130) and can be verified by direct calculations that the expression $\det\|\mathbf{L}\|\mathbf{u}_1^{-1}$ given by Eq. (25) is an invariant of transformation given by Eqs. (131) and (132) for arbitrary spectral parameters. This is an analogue of two-dimensional case (12). The invariance of the noncommutative elements $\mathbf{w}_1\mathbf{w}_2$, $\mathbf{u}_2\mathbf{u}_3$, and $\mathbf{u}_1^{-1}\mathbf{w}_3$ follows from Eqs. (131) and (132) immediately, but the test of the invariance of $\mathbf{H}$ given by Eq. (26) is rather tedious.

4.1.4. Evolutionary lattices. Many three-dimensional extensions of formula (9) have been proposed. In the two-dimensional case, the chain length is the only parameter, while the shape of a lattice and its size both serve as parameters in the three-dimensional case.

Let us consider an auxiliary lattice satisfying the following conditions on the torus:

(1) All the partial triangles of the lattice are oriented as the triangles shown in Fig. 7.

(2) There exists a nontrivial displacement of the lines such that, with regard to the boundary conditions on the torus, the lattice turns into itself.

Such lattices will be referred to as evolutionary lattices. The canonical-transformation operator $\mathfrak{C}$ corresponding to a certain displacement of the lines and formed by local operators $\mathfrak{R}$ given by Eq. (133) will be referred to as an evolution operator [an analogue of Eq. (9)]. By definition, the operator $\mathfrak{R}$ does not affect the minors; hence, the operator $\mathfrak{C}$ does not influence the normalized determinant of the matrix of coefficients, $\mathbf{j}(x, y)$ $\forall\, x, y \in \mathbb{C}$. Therefore, the matrix elements $\mathbf{j}_{\alpha,\beta}$ are integrals of the motion given by the evolution operator $\mathfrak{C}$.

A kagome lattice, which is not considered here, is a basic example of an evolutionary lattice. An evolutionary system on the kagome lattice was formulated in [1, 3].

We now consider the lattice shown in Fig. 2, which is not entirely evolutionary. The upper right part of this figure shows a square $N_2 \times N_3$ lattice, with its vertices numbered as in Fig. 5 with $n_2 = 0, \ldots, N_2 - 1$ and $n_3 = 0, \ldots, N_3 - 1$. The vertices of the auxiliary lattice are numbered as $(2 : n_3)$ and $(3 : n_2)$. In order to distinguish the vertices of the auxiliary lattice from those of the square lattice, we denote the latter by $(1 : n_2, n_3)$. The numbers 1, 2, and 3 in these notations mean the bundle numbers on the auxiliary lattice and correspond to the notation $V = 1, 2, 3$ in Fig. 7. There are N_2N_3 triangles similar to the left triangle in Fig. 7. The operator corresponding to the diagonal translation of the auxiliary line from the lower right cor-

5 In the case of q in general position, the definiteness of the trace is a conditional assumption.

ner to the upper left corner is evidently an ordered product of the local operators $\mathfrak{R}$:

$$\hat{\mathfrak{T}} = \prod_{n_2 = 0\uparrow N_2 - 1} \prod_{n_3 = 0\uparrow N_3 - 1} \mathfrak{R}_{(1\,:\,n_2, n_3), (2\,:\,n_3), (3\,:\,n_2)}, \quad (135)$$

where the ordered product is defined as

$$\prod_{n_j = 0\uparrow N_j - 1} \mathbf{r}_n = \mathbf{r}_0\mathbf{r}_1\mathbf{r}_2\ldots\mathbf{r}_{N_j - 1}. \quad (136)$$

The operator $\hat{\mathfrak{T}}$ is a three-dimensional analogue of the quantum monodromy matrix, with its trace

$$\mathfrak{T} = \mathrm{Trace}_{(2\,:\,n_3), (3\,:\,n_2)}\hat{\mathfrak{T}} \quad (137)$$

over the auxiliary spaces being an analogue of the quantum transfer matrix.[5] The functional $\mathbf{j}(x, y)$ defined above for square lattice (38) generates a complete set of the operators commuting with $\mathfrak{T}$:

$$\mathbf{j}(x, y)\mathfrak{T} = \mathfrak{T}\mathbf{j}(x, y) \quad \forall x, y \in \mathbb{C}. \quad (138)$$

4.1.5. Operator $\mathfrak{Z}$. We have considered above the case of just one auxiliary line producing additional vertices of the type $(2 : n_2)(3 : n_2)$. Let us now consider the case of N_1 auxiliary lines indexed by $n_1 = 0, \ldots, N_1 - 1$. The initial auxiliary line with $n_1 = 0$ is shown in Fig. 8.

The vertices of the original square lattice and those of the auxiliary lattice are numbered as $(1 : n_2, n_3)$ and $(2 : n_1, n_3)$ and $(3 : n_1, n_2)$ (the first, second, and third bundles), respectively. The operator corresponding to the displacement of all the auxiliary lines across the square lattice takes the form

$$\mathfrak{Z}_{1, 2, 3} = \prod_{n_1 = 0\uparrow N_1 - 1} \prod_{n_2 = 0\uparrow N_2 - 1} \prod_{n_3 = 0\uparrow N_3 - 1} \mathfrak{R}_{(1\,:\,n_2, n_3), (2\,:\,n_1, n_3), (3\,:\,n_1, n_2)}. \quad (139)$$

The operator $\mathfrak{Z}_{1, 2, 3}$ implements a rational transformation of the algebra of observables; the resultant will be marked by a star [in place of a prime in definition (131) of $\mathfrak{R}_{1, 2, 3}$]:

$$\mathfrak{Z}_{1, 2, 3}\mathbf{u}_V = \mathbf{u}_V^*\mathfrak{Z}_{1, 2, 3}, \quad \mathfrak{Z}_{1, 2, 3}\mathbf{w}_V = \mathbf{w}_V^*\mathfrak{Z}_{1, 2, 3}, \quad (140)$$
$$V \in \{(1 : n_2, n_3), (2 : n_1, n_3), (3 : n_1, n_2)\}.$$

It is evident that $\mathfrak{Z}_{1, 2, 3}$ represents a cubic lattice. The operator $\hat{\mathfrak{T}}$ is the particular case of $\mathfrak{Z}_{1, 2, 3}$ for $N_1 = 1$; hence, this operator and operator (137) are associated with a layer of the cubic lattice and a layer–layer transfer matrix, respectively.

4.2. Operator $\mathfrak{R}$ in the Finite-Dimensional Case

4.2.1. R matrix. Permutation relations (133) defining the operator $\mathfrak{R}$ are uniquely solvable for finite-dimensional representations (59) and (60) with $q^N = 1$.[6] However, definition (133) slightly changes in the finite-dimensional case.

It follows from definition (133) that, in addition to having a matrix structure, the operator $\mathfrak{R}$ affects the centers of Weyl algebras:

$$\mathfrak{R}_{1, 2, 3}\mathbf{u}_V^N = \mathbf{u}_V'^N\mathfrak{R}_{1, 2, 3}, \quad \mathfrak{R}_{1, 2, 3}\mathbf{w}_V^N = \mathbf{w}_V'^N\mathfrak{R}_{1, 2, 3}. \quad (141)$$

According to relations (131), (132), and $(\mathbf{u} + \mathbf{w})^N = \mathbf{u}^N + \mathbf{w}^N$, this mapping exactly coincides with that for original $\mathbf{u}_W$ and $\mathbf{w}_V$:[7]

$$\begin{aligned} \mathbf{w}_1'^N &= \mathbf{w}_2^N\Lambda_3^N, \quad \mathbf{w}_2'^N = \Lambda_3^{-N}\mathbf{w}_1^N, \quad \mathbf{w}_3'^N = \Lambda_2^{-N}\mathbf{u}_1^{-N}, \\ \mathbf{u}_1'^N &= \Lambda_2^{-N}\mathbf{w}_3^{-N}, \quad \mathbf{u}_2'^N = \Lambda_1^{-N}\mathbf{u}_3^N, \quad \mathbf{u}_3'^N = \mathbf{u}_2^N\Lambda_1^N, \end{aligned} \quad (142)$$

where

$$\begin{aligned} \Lambda_1^N &= \mathbf{u}_1^{-N}\mathbf{u}_3^N + \mathbf{u}_1^{-N}\mathbf{w}_1^N + \kappa_1^N\mathbf{w}_1^N\mathbf{u}_2^{-N}, \\ \Lambda_2^N &= \frac{\kappa_1^N}{\kappa_2^N}\mathbf{u}_2^{-N}\mathbf{w}_3^{-N} + \frac{\kappa_3^N}{\kappa_2^N}\mathbf{u}_1^{-N}\mathbf{w}_2^{-N} + \frac{\kappa_1^N\kappa_3^N}{\kappa_2^N}\mathbf{u}_2^{-N}\mathbf{w}_2^{-N}, \\ \Lambda_3^N &= \mathbf{w}_1^N\mathbf{w}_3^{-N} + \mathbf{u}_3^N\mathbf{w}_3^{-N} + \kappa_3^N\mathbf{w}_2^{-N}\mathbf{u}_3^N. \end{aligned} \quad (143)$$

We now define

$$u_V' = \sqrt[N]{\mathbf{u}_V'^N}, \quad w_V' = \sqrt[N]{\mathbf{w}_V'^N}, \quad V = 1, 2, 3, \quad (144)$$

and impose the three natural restrictions on the phases:

$$w_1 w_2 = w_1' w_2', \quad u_1^{-1} w_3 = u_1'^{-1} w_3', \quad u_2 u_3 = u_2' u_3'. \quad (145)$$

Let

$$\mathbf{x}_V' = \frac{\mathbf{u}_V'}{u_V'}, \quad \mathbf{z}_V' = \frac{\mathbf{w}_V'}{w_V'}, \quad V = 1, 2, 3. \quad (146)$$

Because $\mathfrak{R}$ is an adequately normalized canonical mapping, there exists (according to the Schur lemma) a

[5] In the case of q in general position, the definiteness of the trace is a conditional assumption.

[6] One more case of the unique solvability in terms of nonlocal quantum dilogarithms is the modular dualization of Weyl algebras by the Faddeev method in the strong-coupling regime with the extra requirement that the operator R should be unitary.

[7] In terms of the formal modular dualization, the Nth powers $\mathbf{u}_V^N$ and $\mathbf{w}_V^N$ are modular partners of $\mathbf{u}_V$ and $\mathbf{w}_V$.

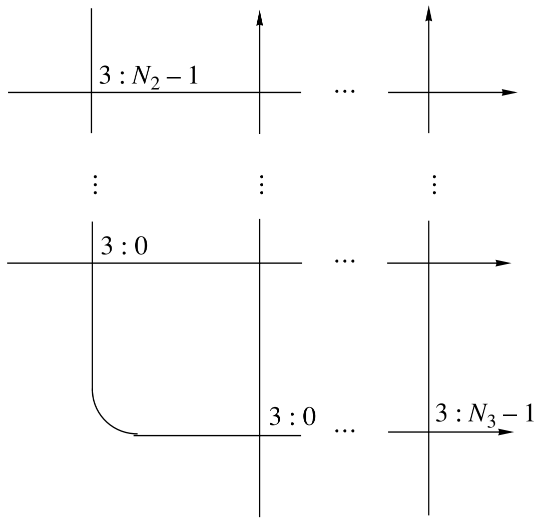

Fig. 8. An auxiliary map of the square lattice.

unique (up to a common factor) $N^3 \times N^3$ matrix $\mathbf{R}_{1,2,3}$ such that

$$\mathbf{x}'_V\mathbf{R}_{1,2,3} = \mathbf{R}_{1,2,3}\mathbf{x}_V, \quad \mathbf{z}'_V\mathbf{R}_{1,2,3} = \mathbf{R}_{1,2,3}\mathbf{z}_V, \quad V = 1, 2, 3. \tag{147}$$

In order to write out the matrix elements of $\mathbf{R}$ in basis (58), we introduce some extra notations.

Let p be a point of a Fermat curve $\mathcal{F}$:

$$p \stackrel{\text{def}}{=} (x, y) \in \mathcal{F} \Leftrightarrow x^N + y^N = 1. \tag{148}$$

We define a function $W_p(n)$, with $p \in \mathcal{F}$ and $n \in \mathbb{Z}_N$, satisfying the equations

$$\frac{W_p(n)}{W_p(n-1)} = \frac{y}{1 - xq^n}, \quad W_p(0) = 1. \tag{149}$$

The function $W_p(n)$ (which is referred to as a "q-gamma function," "q-exponential function," or "q dilogarithm") has a series of remarkable properties at the root of unity. An introduction to "q root of unity" hypergeometric series can be found, e.g., in [21].

We then define the $N^3 \times N^3$ matrix $\mathbf{R}_{1,2,3}$ as a matrix-valued function of four points, p_1, p_2, p_3, and p_4, of the Fermat curve:

$$\begin{gathered}\langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R} | \sigma'_1, \sigma'_2, \sigma'_3 \rangle \stackrel{\text{def}}{=} R^{\sigma'_1, \sigma'_2, \sigma'_3}_{\sigma_1, \sigma_2, \sigma_3} \\ = \delta_{\sigma_2 + \sigma_3, \sigma'_2 + \sigma'_3} q^{(\sigma'_1 - \sigma_1)\sigma'_3} \frac{W_{p_1}(\sigma_2 - \sigma_1) W_{p_2}(\sigma'_2 - \sigma'_1)}{W_{p_3}(\sigma'_2 - \sigma_1) W_{p_4}(\sigma_2 - \sigma'_1)},\end{gathered} \tag{150}$$

where the x coordinates of the four points are related by the equation

$$x_1 x_2 = q x_3 x_4. \tag{151}$$

This matrix, referred to as an $\mathbf{R}$ matrix of the Zamolodchikov–Bazhanov–Baxter model [16, 21], implements transformation (147) in basis (58) in the variables

$$\begin{gathered} x_1 = \frac{q^{-1/2}}{\kappa_1}\frac{u_2}{u_1}, \quad x_2 = q^{-1/2}\kappa_2\frac{u'_2}{u'_1}, \\ x_3 = q^{-1}\frac{u'_2}{u_1}, \quad x_4 = q^{-1}\frac{\kappa_2 u_2}{\kappa_1 u'_1}, \end{gathered} \tag{152}$$

and

$$\begin{gathered} \frac{y_3}{y_1} = \kappa_1\frac{w_1}{u'_3}, \quad \frac{y_4}{y_1} = q^{-1/2}\kappa_3\frac{w_3}{w_2}, \\ \frac{y_3}{y_2} = \frac{w'_2}{w_3}, \quad \frac{y_4}{y_2} = q^{-1/2}\frac{\kappa_3 u'_3}{\kappa_1 w'_1}, \end{gathered} \tag{153}$$

where u'_V, w'_V, u_V, and w_V are related by Eqs. (142), (144), and (145). To prove statement (147), one should evaluate matrix elements of each relation in (147) and use definitions (58) and (149).

$\mathbf{R}$ matrix (150) is a matrix function of three continuous parameters [see Eq. (151)] and three discrete parameters related to the points y_j of the Fermat curves: $y_j^N = 1 - x_j^N$. There are just three discrete parameters because the simultaneous change of phases of all the variables y_j, $y_j \mapsto qy_j$, does not influence matrix elements (150). On the other hand, parametrization (152) and (153) has nine independent variables (u_V, w_V, κ_V) and, by virtue of Eq. (145), just three independent phases in the set (u'_V, w'_V). The set of Eqs. (152) and (153) will be referred to as a free parametrization of the $\mathbf{R}$ matrix. The notations $\mathbf{R}(p_1, p_2, p_3, p_4)$ and $\mathbf{R}(u_V, w_V, \kappa_V)$ both are used in what follows.

There is an important distinction between the mapping $\mathfrak{R}$ and the similarity transformation by the $\mathbf{R}$ matrix. According to definition (133), the relation

$$\mathfrak{R}F(\mathbf{u}_V, \mathbf{w}_V) = F(\mathbf{u}'_V, \mathbf{w}'_V)\mathfrak{R}, \quad V = 1, 2, 3, \tag{154}$$

is valid for an arbitrary rational function $F(\mathbf{u}_V, \mathbf{w}_V)$ on the Weyl algebra. On the other hand,

$$\mathbf{R}F(u_V\mathbf{x}_V, w_V\mathbf{z}_V) = F(u_V\mathbf{x}'_V, w_V\mathbf{z}'_V)\mathbf{R}, \tag{155}$$

at the root of unity, when $\mathbf{u}_V = u_V\mathbf{x}_V$ and $\mathbf{w}_V = w_V\mathbf{z}_V$. In order words, the mapping $\mathfrak{R}$ is equivalent to the similarity transformation by the $\mathbf{R}$ matrix and, in addition, the transformation ($u_V \mapsto u'_V$, $w_V \mapsto w'_V$) of the centers of Weyl algebra. Such a property has been noted in [22].

4.2.2. Evolutionary mappings in the finite-dimensional case. When constructing finite-dimensional (i.e., matrix) parts of the evolutionary mappings $\mathfrak{T}$ and $\mathfrak{Z}$, we have to consider iterations of the functional mappings (u_V, w_V) $\mapsto$ (u'_V, w'_V). This causes some complications in the analysis of finite-dimensional operators. Let $\mathfrak{Z}$ be an evolutionary mapping constructed from different $\mathfrak{R}$ and defined on an auxiliary lattice:

$$\mathfrak{Z}F(\mathbf{u}_V, \mathbf{w}_V) = F(\mathbf{u}_V^*, \mathbf{w}_V^*)\mathfrak{Z}. \tag{156}$$

The mapping $\mathfrak{Z}$ naturally decomposes into the finite-dimensional part $\mathbf{Z}$,

$$\mathbf{Z}F(u_V\mathbf{x}_V, w_V\mathbf{z}_V) = F(u_V\mathbf{x}_V^*, w_V\mathbf{z}_V^*)\mathbf{Z}, \tag{157}$$

with

$$\mathbf{x}_V^* = \frac{\mathbf{u}_V^*}{\sqrt[N]{\mathbf{u}_V^{*N}}}, \quad \mathbf{z}_V^* = \frac{\mathbf{w}_V^*}{\sqrt[N]{\mathbf{w}_V^{*N}}}, \tag{158}$$

and the functional part

$$u_V \mapsto u_V^* = \sqrt[N]{\mathbf{u}_V^{*N}}, \quad w_V \mapsto w_V^* = \sqrt[N]{\mathbf{w}_V^{*N}}. \tag{159}$$

There are two specific functions among all functions on the algebra of observables, namely, the functionals $\mathbf{j}(x, y|\mathbf{u}_V, \mathbf{w}_V)$ and $J(x^N, y^N|\mathbf{u}_V^N, \mathbf{w}_V^N)$ generating the integrals of the motion defined by the evolutionary mapping $\mathfrak{Z}$. They both are invariants of the mapping $\mathfrak{Z}$:

$$\begin{aligned}\mathbf{j}(x, y|\mathbf{u}_V, \mathbf{w}_V) &= \mathbf{j}(x, y|\mathbf{u}_V^*, \mathbf{w}_V^*),\\ \mathbf{j}(x^N, y^N|\mathbf{u}_V^N, \mathbf{w}_V^N) &= \mathbf{j}(x^N, y^N|\mathbf{u}_V^{*N}, \mathbf{w}_V^{*N}).\end{aligned} \tag{160}$$

According to definition (157), the matrix $\mathbf{j}$ is not an invariant of $\mathbf{Z}$:

$$\begin{aligned}\mathbf{j}(x, y|u_V\mathbf{x}_V, w_V\mathbf{z}_V) &\equiv \mathbf{j}(x, y|u_V^*\mathbf{x}_V^*, w_V^*\mathbf{z}_V^*)\\ &= \mathbf{Z}\mathbf{j}(x, y|u_V^*\mathbf{x}_V, w_V^*\mathbf{z}_V)\mathbf{Z}^{-1}.\end{aligned} \tag{161}$$

Therefore, if the set (u_V, w_V, κ_V) is in general position, so that mapping (159) is nontrivial, then mapping (159) is an isospectral deformation of the operator $\mathbf{j}(x, y)$ and the matrix $\mathbf{Z}$ implements this deformation.

When constructing the hypothesis on quantum separation of variables, we will use the method of isospectral deformations.

A submanifold of initial values of (u_V, w_V, κ_V) can be chosen (by far not uniquely) such that

$$u_V^* = u_V, \quad w_V^* = w_V. \tag{162}$$

Under such an additional condition, the isospectral mapping becomes trivial and $\mathbf{j}(x, y)$ generates a complete set of independent matrices commuting with $\mathbf{Z}$. This means that the problem of finding the spectrum of $\mathbf{Z}$ is solvable. Therefore, cyclic boundary conditions (162) are used for formulating exactly solvable models of statistical mechanics on three-dimensional lattices. It is noteworthy that the problem of the simultaneous parametrization of (u_V, w_V) and (u_V^*, w_V^*) under periodic boundary conditions are completely solvable within the framework of algebraic geometry (as was mentioned above).

It is useful to comment upon the particular case when the centers of Weyl algebra belong to a specific submanifold such that

$$u'_V = u_V, \quad w'_V = w_V \tag{163}$$

for each $\mathbf{R}$ matrix. This condition is stronger than periodic boundary condition (162) for the whole evolutionary mapping discussed above. There are three independent equations in system (163) for an individual $\mathbf{R}$ matrix. This manifold can be parametrized in terms of the points p_1, p_2, p_3, and p_4:

$$\frac{w_3}{w_2} = \frac{y_2}{y_3}, \quad \frac{w_1}{u_3} = q^{-1/2}\frac{x_1y_3}{x_3y_1}, \quad \frac{u_1}{u_2} = \frac{1}{qx_3}, \tag{164}$$

$$\kappa_1 = q^{1/2}\frac{x_3}{x_1}, \quad \kappa_2 = q^{1/2}\frac{x_4}{x_1}, \quad \kappa_3 = q^{1/2}\frac{y_3y_4}{y_1y_2}. \tag{165}$$

In particular, the finite-dimensional tetrahedron equation for $\mathfrak{R}$ matrices, which is an immediate consequence of Eq. (134), is defined on this submanifold, on which the operator $\mathbf{R}$ is completely equivalent to the similarity transformation by the $\mathbf{R}$ matrix [15, 16, 21]. The four $\mathbf{R}$ matrices entering into the tetrahedron equation have different arguments, and the famous tetrahedral bound naturally originates as a consequence of Eqs. (164) for these $\mathbf{R}$ matrices. The tetrahedron equation is the integrability condition for the Zamolodchikov–Bazhanov–Baxter model, and the Zamolodchikov parametrization in terms of the dihedral angles θ_1, θ_2, and θ_3 of a spherical triangle is equivalent to the equations

$$\begin{aligned}\kappa_1^N &= \left(\tan\frac{\theta_1}{2}\right)^2, \quad \kappa_2^N = \left(\cot\frac{\theta_2}{2}\right)^2,\\ \kappa_3^N &= \left(\tan\frac{\theta_3}{2}\right)^2.\end{aligned} \tag{166}$$

There exists one more parametrization in terms of the generatrices u_j, w_j, and κ_j, which is equivalent to that described above. The parametrization is defined by projecting the spherical triangle onto a plane and has the form

$$\begin{aligned}\kappa_1^N &= -\frac{(Y-Z')(Y'-Z)}{(Y-Z)(Y'-Z')},\\ \kappa_2^N &= -\frac{(X-Z)(X'-Z')}{(X'-Z)(X-Z')},\\ \kappa_3^N &= -\frac{(X'-Y)(X-Y')}{(X-Y)(X'-Y')},\end{aligned} \tag{167}$$

so that

$$\frac{w_3^N}{w_2^N} = \frac{(X'-Y')(X-Z)}{(X-Y')(X'-Z)},$$
$$\frac{w_1^N}{u_3^N} = -\frac{(Y-Z)(X-Y')}{(Y'-Z)(X-Y)}, \qquad (168)$$
$$\frac{u_1^N}{u_2^N} = \frac{(Y'-Z')(X-Z)}{(Y'-Z)(X-Z')}.$$

By virtue of the Møbius invariance of cross ratios, this parametrization with six complex-valued points has only three independent parameters. One of the advantages of the parametrization with cross ratios is that the geometrical tetrahedral condition is always satisfied in the case of the tetrahedron equation. Such a parametrization with cross ratios was used above in Eq. (89) for square nonuniform lattices.

In a more general approach, condition (163) is not imposed separately on each **R** matrix. In this case, the integrability is due to the principle of isospectral deformations (161) and cyclic boundary conditions (162) rather than to the tetrahedron equation (although it can be derived).

5. QUANTUM TRANSFER MATRIX

In this section, we explicitly construct a finite-dimensional representation of the operators $\mathfrak{Z}_{1,2,3}$ and $\mathfrak{T}$ entering into Eqs. (135), (137), and (139). The finite-dimensional representation of operator (139) is the statistical sum for a cubic lattice under open boundary conditions. In this case, the most complicated task is to allow for the evolution of the numerical parameters $\mathbf{u}_V^N$ and $\mathbf{w}_V^N$ under sequential mappings $\mathfrak{R}$.

5.1. General Position

To evaluate matrix elements of operator $\mathfrak{Z}_{1,2,3}$ (139), one should evidently use Eqs. (141), (146), and (147) in sequence for all the operators $\mathfrak{R}_{1,2,3}$ entering into $\mathfrak{Z}_{1,2,3}$.

In the notations of Eq. (139), the original parameters of the local Weyl algebra for $\mathfrak{Z}_{1,2,3}$ are $(u_{1:n_2,n_3}, w_{1:n_2,n_3}, \kappa_{1:n_2,n_3})$ for the basic square lattice; for the vertices on auxiliary lines, they are $(u_{2:n_1,n_3}, w_{2:n_1,n_3}, \kappa_{2:n_1,n_3})$ and $(u_{3:n_1,n_2}, w_{3:n_1,n_2}, \kappa_{3:n_1,n_2})$. Here, the index j in the notation $V = (j:n_k, n_l)$ numbers the "bundles" of the auxiliary lattice for the left triangle in Fig. 7 and the indices n_k and n_l number a vertex in the corresponding square lattice, which are basic if $j = 1$ and two auxiliary if $j = 2, 3$. Formulas (142) and (143) after scrupulous comparison with the structure $\mathbf{Z}_{1,2,3}$ (139) define a three-dimensional recursion on a cubic lattice:

$$\begin{aligned} w_{1,\mathbf{n}+\mathbf{e}_1}^N &= w_{2,\mathbf{n}}^N \Lambda_{3,\mathbf{n}}^N, \quad & w_{2,\mathbf{n}+\mathbf{e}_2}^N &= \Lambda_{3,\mathbf{n}}^{-N} w_{1,\mathbf{n}}^N, \\ w_{3,\mathbf{n}+\mathbf{e}_3}^N &= \Lambda_{2,\mathbf{n}}^{-N} u_{1,\mathbf{n}}^{-N}, \quad & u_{1,\mathbf{n}+\mathbf{e}_1}^N &= \Lambda_{2,\mathbf{n}}^{-N} w_{3,\mathbf{n}}^{-N}, \\ u_{2,\mathbf{n}+\mathbf{e}_2}^N &= \Lambda_{1,\mathbf{n}}^{-N} u_{3,\mathbf{n}}^N, \quad & u_{3,\mathbf{n}+\mathbf{e}_3}^N &= u_{2,\mathbf{n}}^N \Lambda_{1,\mathbf{n}}^N, \end{aligned} \qquad (169)$$

where

$$\begin{aligned} \Lambda_{1,\mathbf{n}}^N &= u_{1,\mathbf{n}}^{-N} u_{3,\mathbf{n}}^N + u_{1,\mathbf{n}}^{-N} w_{1,\mathbf{n}}^N + \kappa_{1,\mathbf{n}}^N w_{1,\mathbf{n}}^N u_{2,\mathbf{n}}^{-N}, \\ \Lambda_{2,\mathbf{n}}^N &= \frac{\kappa_{1,\mathbf{n}}^N}{\kappa_{2,\mathbf{n}}^N} u_{2,\mathbf{n}}^{-N} w_{3,\mathbf{n}}^{-N} + \frac{\kappa_{3,\mathbf{n}}^N}{\kappa_{2,\mathbf{n}}^N} u_{1,\mathbf{n}}^{-N} w_{2,\mathbf{n}}^{-N} \\ &\quad + \frac{\kappa_{1,\mathbf{n}}^N \kappa_{3,\mathbf{n}}^N}{\kappa_{2,\mathbf{n}}^N} u_{2,\mathbf{n}}^{-N} w_{2,\mathbf{n}}^{-N}, \\ \Lambda_{3,\mathbf{n}}^N &= w_{1,\mathbf{n}}^N w_{3,\mathbf{n}}^{-N} + u_{3,\mathbf{n}}^N w_{3,\mathbf{n}}^{-N} + \kappa_{3,\mathbf{n}}^N w_{2,\mathbf{n}}^{-N} u_{3,\mathbf{n}}^N. \end{aligned} \qquad (170)$$

Here, the three-dimensional e vectors

$$\mathbf{n} = (n_1, n_2, n_3) = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \qquad (171)$$

define the vertices of the cubic lattice. The κ parameters and the initial conditions for this recursion are given by the relations

$$\begin{gathered} u_{1,n_2\mathbf{e}_2+n_3\mathbf{e}_3} = u_{1:n_2,n_3}, \\ w_{1,n_2\mathbf{e}_2+n_3\mathbf{e}_3} = w_{1:n_2,n_3}, \quad \kappa_{1,\mathbf{n}} = \kappa_{1:n_2,n_3}, \\ u_{2,n_1\mathbf{e}_1+n_3\mathbf{e}_3} = u_{2:n_1,n_3}, \\ w_{2,n_1\mathbf{e}_1+n_3\mathbf{e}_3} = w_{2:n_1,n_3}, \quad \kappa_{2,\mathbf{n}} = \kappa_{2:n_1,n_3}, \\ u_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2} = u_{3:n_1,n_2}, \\ w_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2} = w_{3:n_1,n_2}, \quad \kappa_{3,\mathbf{n}} = \kappa_{3:n_1,n_2}. \end{gathered} \qquad (172)$$

The Nth root of expression (169) is defined by Eq. (145), which takes the following form on the lattice:

$$\begin{aligned} w_{1,\mathbf{n}+\mathbf{e}_1} w_{2,\mathbf{n}+\mathbf{e}_2} &= w_{1,\mathbf{n}} w_{2,\mathbf{n}}, \\ u_{1,\mathbf{n}+\mathbf{e}_1}^{-1} w_{3,\mathbf{n}+\mathbf{e}_3} &= u_{1,\mathbf{n}}^{-1} w_{3,\mathbf{n}}, \\ u_{2,\mathbf{n}+\mathbf{e}_2} u_{3,\mathbf{n}+\mathbf{e}_3} &= u_{2,\mathbf{n}} u_{3,\mathbf{n}}. \end{aligned} \qquad (173)$$

The action of the operator $\mathfrak{Z}_{1,2,3}$ on the parameters $\mathbf{u}_V^N \equiv u_V^N$ and $\mathbf{w}_V^N \equiv w_V^N$ [see Eq. (140)],

$$\begin{aligned} \mathfrak{Z}_{1,2,3}\mathbf{u}_V^N &= \mathbf{u}_V^{*N}\mathfrak{Z}_{1,2,3}, \\ \mathfrak{Z}_{1,2,3}\mathbf{w}_V^N &= \mathbf{w}_V^{*N}\mathfrak{Z}_{1,2,3}, \end{aligned} \qquad (174)$$

is expressed in terms of recursion (169) and (170) as

$$\mathbf{u}_{1:n_2,n_3}^{*N} = u_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}^N,$$
$$\mathbf{w}_{1:n_2,n_3}^{*N} = w_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}^N,$$

$$\begin{aligned}\mathbf{u}^{*N}_{2:n_1,n_3} &= u^N_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3},\\ \mathbf{w}^{*N}_{2:n_1,n_3} &= w^N_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3},\\ \mathbf{u}^{*N}_{3:n_1,n_2} &= u^N_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3},\\ \mathbf{w}^{*N}_{3:n_1,n_2} &= w^N_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}.\end{aligned} \tag{175}$$

According to Eq. (158), one should introduce the normalized Weyl elements $\mathbf{x}^*_V = \frac{\mathbf{u}^*_V}{u^*_V}$ and $\mathbf{z}^*_V = \frac{\mathbf{w}^*_V}{w^*_V}$ and then define the unique matrix $\mathbf{Z}_{1,2,3}$

$$\mathbf{Z}_{1,2,3}\mathbf{x}_V = \mathbf{x}^*_V\mathbf{Z}_{1,2,3}, \quad \mathbf{Z}_{1,2,3}\mathbf{z}_V = \mathbf{z}^*_V\mathbf{Z}_{1,2,3}, \tag{176}$$

with the matrix elements determined by Eq. (150):

$$\langle\sigma_1,\sigma_2,\sigma_3|\mathbf{Z}_{1,2,3}|\sigma_1^*,\sigma_2^*,\sigma_3^*\rangle = \sum_{\sigma_1,\sigma_2,\sigma_3}\prod_{n_1,n_2,n_3}\langle\sigma_{1,\mathbf{n}},\sigma_{2,\mathbf{n}},\sigma_{3,\mathbf{n}}|\mathbf{R}_\mathbf{n}|\sigma_{1,\mathbf{n}+\mathbf{e}_1},\sigma_{2,\mathbf{n}+\mathbf{e}_2},\sigma_{3,\mathbf{n}+\mathbf{e}_3}\rangle. \tag{177}$$

Here, we use the vectors of indices:

$$\begin{aligned}\sigma_1 &= \{\sigma_{1,n_2\mathbf{e}_2+n_3\mathbf{e}_3}\}, \quad \sigma_2 = \{\sigma_{2,n_1\mathbf{e}_1+n_3\mathbf{e}_3}\},\\ \sigma_3 &= \{\sigma_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2}\}, \quad n_j = 0,\ldots,N_j-1,\end{aligned} \tag{178}$$

$$\begin{aligned}\sigma_1^* &= \{\sigma_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}\}, \quad \sigma_2^* = \{\sigma_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3}\},\\ \sigma_3^* &= \{\sigma_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}\}, \quad n_j = 0,\ldots,N_j-1,\end{aligned} \tag{179}$$

where the summation is extended over all inner $\sigma_{j,\mathbf{n}}$, with $0 < n_j < N_j$. Each matrix $\mathbf{R}_\mathbf{n}$ entering into Eq. (177) is a matrix function (150) of variables $p_{1,\mathbf{n}}$, $p_{2,\mathbf{n}}$, $p_{3,\mathbf{n}}$, and $p_{4,\mathbf{n}}$. According to Eqs. (152), (153), and (169), they take the form

$$\begin{aligned}x_{1,\mathbf{n}} &= \frac{q^{-1/2}u_{2,\mathbf{n}}}{\kappa_{1,\mathbf{n}}u_{1,\mathbf{n}}}, \quad x_{2,\mathbf{n}} = q^{-1/2}\kappa_{2,\mathbf{n}}\frac{u_{2,\mathbf{n}+\mathbf{e}_2}}{u_{1,\mathbf{n}+\mathbf{e}_1}},\\ x_{3,\mathbf{n}} &= q^{-1}\frac{u_{2,\mathbf{n}+\mathbf{e}_2}}{u_{1,\mathbf{n}}}, \quad x_{4,\mathbf{n}} = q^{-1}\frac{\kappa_{2,\mathbf{n}}}{\kappa_{1,\mathbf{n}}}\frac{u_{2,\mathbf{n}}}{u_{1,\mathbf{n}+\mathbf{e}_1}},\end{aligned} \tag{180}$$

$$\begin{aligned}\frac{y_{3,\mathbf{n}}}{y_{1,\mathbf{n}}} &= \kappa_{1,\mathbf{n}}\frac{w_{1,\mathbf{n}}}{u_{3,\mathbf{n}+\mathbf{e}_3}}, \quad \frac{y_{4,\mathbf{n}}}{y_{1,\mathbf{n}}} = q^{-1/2}\kappa_{3,\mathbf{n}}\frac{w_{3,\mathbf{n}}}{w_{2,\mathbf{n}}},\\ \frac{y_{3,\mathbf{n}}}{y_{2,\mathbf{n}}} &= \frac{w_{2,\mathbf{n}+\mathbf{e}_2}}{w_{3,\mathbf{n}}}, \quad \frac{y_{4,\mathbf{n}}}{y_{2,\mathbf{n}}} = q^{-1/2}\frac{\kappa_{3,\mathbf{n}}}{\kappa_{1,\mathbf{n}}}\frac{u_{3,\mathbf{n}+\mathbf{e}_3}}{w_{1,\mathbf{n}+\mathbf{e}_1}}.\end{aligned} \tag{181}$$

In addition to the open system defined by recursion (169) and (170) and to the case of complete cyclic boundary conditions, partial cyclic boundary conditions can be considered. Cyclic boundary conditions imposed on the second and third bundles are an important example:

$$\begin{aligned}u^*_{2:n_1,n_3} &= u_{2:n_1,n_3}, \quad w^*_{2:n_1,n_3} = w_{2:n_1,n_3},\\ u^*_{3:n_1,n_2} &= u_{3:n_1,n_3}, \quad w^*_{3:n_1,n_2} = w_{3:n_1,n_3}.\end{aligned} \tag{182}$$

These boundary conditions imply the evaluation of the trace, $\mathbf{Z}_1 = \mathrm{Trace}_{2,3}\mathbf{Z}_{1,2,3}$:

$$\langle\sigma_1|\mathbf{Z}_1|\sigma_1^*\rangle = \sum_{\sigma_2,\sigma_3}\langle\sigma_1,\sigma_2,\sigma_3|\mathbf{Z}_{1,2,3}|\sigma_1^*,\sigma_2,\sigma_3\rangle. \tag{183}$$

In general, the periodicity of the first bundle does not follow from periodic conditions (182). The operator $\mathbf{Z}_1$ acts only on the first bundle, where the auxiliary lattice is a square one. This means that $\mathbf{Z}_1$ satisfies relations similar to Eq. (161) for the functional $\mathbf{j}(x, y)$ on the square lattice considered in Section 2.3.3 (with the evident change of the notations: $\mathbf{u}_{n_2,n_3} \mapsto \mathbf{u}_{1:n_2,n_3}$, $\mathbf{w}_{n_2,n_3} \mapsto \mathbf{w}_{1:n_2,n_3}$, and $\kappa_{n_2,n_3} \mapsto \kappa_{1:n_2,n_3}$). It is useful to take the parameters $u_{1:n_2,n_3}$ and $w_{1:n_2,n_3}$ as arguments of $\mathbf{j}$, i.e.,

$$\mathbf{j}(x, y) = \mathbf{j}(x, y|\{u_{1:n_2,n_3}, w_{1:n_2,n_3}\}). \tag{184}$$

In this case, the matrix $\mathbf{Z}_1$ defines the isospectral transformation

$$\begin{aligned}&\mathbf{Z}_1\mathbf{j}(x, y|\{u_{1:n_2,n_3}, w_{1:n_2,n_3}\})\\ &= \mathbf{j}(x, y|\{u^*_{1:n_2,n_3}, w^*_{1:n_2,n_3}\})\mathbf{Z}_1.\end{aligned} \tag{185}$$

If the variables $u_{j,\mathbf{n}}$ and $w_{j,\mathbf{n}}$ satisfy conditions (182) and, in addition, are periodic in the first direction, then relation (185) reduces to $\mathbf{Z}_1\mathbf{j}(x, y) = \mathbf{j}(x, y)\mathbf{Z}_1$, i.e., the operator-valued polynomials generated by $\mathbf{j}$ are integrals of motion for the finite-dimensional $\mathbf{Z}_1$. It is worth noting that the functionals $\mathbf{j}$ generating the integrals of motion can be constructed in the same way in the auxiliary planes corresponding to the second and third bundles. In this sense, the three-dimensional invariance of our approach actually takes place.

5.2. Soliton Solution

Recursion (169) and (170) under initial conditions (172) is a system of the equations of motion for the classical integrable model on the cubic lattice. As was mentioned above, this system can be solved completely within the framework of algebraic geometry (see [3, 11]). Equations (180) and (181) with the parametrization $u^N_{j,\mathbf{n}}$, $w^N_{j,\mathbf{n}}$, and $\kappa^N_{j,\mathbf{n}}$, written out in terms of ratios of the Θ functions and prime forms in [11], explicitly defines the parameters of all the $\mathbf{R}_\mathbf{n}$ matrices in Eq. (177). In this case, periodic boundary conditions (162) determine an algebraic curve, with the Θ functions defined on the Jacobian of this curve. For example, under partial periodic boundary conditions (182), the algebraic curve is defined by the equation $J(x^N, y^N) = 0$, where J is the determinant of the $\mathbb{C}$-number matrix of coefficients for the square lattice of the first bundle [see Eq. (71) et seq.]. However, since expressions of algebraic geometry are not very suitable in practice, we consider only the limiting case when the spectral curve degenerates into a sphere with distinguished points. In this case, the parametrization $(u^N_{j,\mathbf{n}}, w^N_{j,\mathbf{n}})$ is treated as a soliton sector of solutions of the equations.

In order to parameterize $\mathbf{R}_\mathbf{n}$, we need to evaluate the quantities $(u_{j,\mathbf{n}}, w_{j,\mathbf{n}}, \kappa_{j,\mathbf{n}})$ rather than their Nth powers; hence, we will use the Nth roots of the rational expressions. Let complex numbers X_{n_1}, X'_{n_1}, Y_{n_1}, Y'_{n_2}, Z_{n_3}, and Z'_{n_3} be points in general position of an $N_1 \times N_2 \times N_3$ cubic lattice, with the roots

$$e(X, Y) : e(X, Y)^N = X - Y \tag{186}$$

chosen for different pairs of these numbers. We introduce [see Eq. (167)]

$$\begin{aligned}
\kappa_{1,\mathbf{n}} &= q^{1/2}\frac{e(Y_{n_2}, Z'_{n_3})e(Y'_{n_2}, Z_{n_3})}{e(Y_{n_2}, Z_{n_3})e(X'_{n_2}, Z'_{n_3})},\\
\kappa_{2,\mathbf{n}} &= q^{1/2}\frac{e(X_{n_1}, Z_{n_3})e(X'_{n_1}, Z'_{n_3})}{e(X'_{n_1}, Z_{n_3})e(X_{n_1}, Z'_{n_3})},\\
\kappa_{3,\mathbf{n}} &= q^{1/2}\frac{e(X'_{n_1}, Y_{n_2})e(X_{n_1}, Y'_{n_2})}{e(X_{n_1}, Y_{n_2})e(X'_{n_1}, Y'_{n_2})}.
\end{aligned} \tag{187}$$

The substitution

$$\begin{aligned}
u_{1,\mathbf{n}} &= -q^{-1/2}\xi_{3,n_3}\frac{e(Y'_{n_2}, Z'_{n_3})}{e(Y'_{n_2}, Z_{n_3})}\frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}},\\
w_{1,\mathbf{n}} &= -\xi_{2,n_2}\frac{e(Y_{n_2}, Z_{n_3})}{e(Y'_{n_2}, Z_{n_3})}\frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}},\\
u_{2,\mathbf{n}} &= -q^{-1/2}\xi_{3,n_3}\frac{e(X_{n_1}, Z'_{n_3})}{e(X_{n_1}, Z_{n_3})}\frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}},\\
w_{2,\mathbf{n}} &= -\xi_{1,n_1}\frac{e(X'_{n_1}, Z_{n_3})}{e(X_{n_1}, Z_{n_3})}\frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}},\\
u_{3,\mathbf{n}} &= -q^{-1/2}\xi_{2,n_2}\frac{e(X_{n_1}, Y_{n_2})}{e(X_{n_1}, Y'_{n_2})}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}},\\
w_{3,\mathbf{n}} &= -\xi_{1,n_1}\frac{e(X'_{n_1}, Y'_{n_2})}{e(X_{n_1}, Y'_{n_2})}\frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}
\end{aligned} \tag{188}$$

is a change of variables in equations of motion (169) and (170) such that $\tau^N_{j,\mathbf{n}}$ are Lagrangian coordinates. We solve the Lagrange equations in terms of the Casoratti determinant, namely, the function $H^{(g)}$ of the g-dimensional vector $(f_0, f_1, \ldots, f_{g-1})$ with the parameters $(P_0, P_1, \ldots, P_{g-1})$:

$$H^{(g)}(\{f_k\}_{k=0}^{g-1}) \overset{\text{def}}{=} \frac{\det\left|P_j^i - f_j P_j'^i\right|_{i,j=0}^{g-1}}{\prod_{i>j}(P_i - P_j)}. \tag{189}$$

Let

$$\sigma_k(X) \overset{\text{def}}{=} \frac{P'_k - X}{P_k - X}. \tag{190}$$

Equations (169) and (170) have the local solution

$$\begin{aligned}
\tau^N_{1,\mathbf{n}} &= H^{(g)}\left(\left\{f_k\frac{I_k(\mathbf{n})}{\sigma_k(X_{n_1})}\right\}_{k=0}^{g-1}\right),\\
\tau^N_{2,\mathbf{n}} &= H^{(g)}\left(\left\{f_k\frac{I_k(\mathbf{n})}{\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right),\\
\tau^N_{3,\mathbf{n}} &= H^{(g)}\left(\left\{f_k\frac{I_k(\mathbf{n})}{\sigma_k(Z_{n_3})}\right\}_{k=0}^{g-1}\right),\\
\lambda^N_{\mathbf{n}} &= H^{(g)}\left(\left\{f_k\frac{I_k(\mathbf{n})\sigma_k(Z'_{n_3})}{\sigma_k(X_{n_1})\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right),
\end{aligned} \tag{191}$$

where

$$\begin{aligned}
I_n(\mathbf{n}) = {} & \left(\prod_{j_1=0}^{n_1-1}\frac{\sigma_k(X'_{j_1})}{\sigma_k(X_{j_1})}\right)\left(\prod_{j_2=0}^{n_2-1}\frac{\sigma_k(Y'_{j_2})}{\sigma_k(Y_{j_2})}\right)\\
& \times\left(\prod_{j_3=0}^{n_3-1}\frac{\sigma_k(Z'_{j_3})}{\sigma_k(Z_{j_3})}\right).
\end{aligned} \tag{192}$$

The expression for $\lambda^N_{\mathbf{n}}$ is to be used below in Eqs. (193) for $y_{j,\mathbf{n}}$. The arguments of the functions in Eqs. (191) contain an arbitrary g-dimensional complex vector $(f_0, f_1, \ldots, f_{g-1})$. Expressions (191) represent a local solution of Eqs. (169) and (170) for arbitrary parameters, including the number g of soliton modes. From the point of view of algebraic geometry, the vector $\{f_k\}_{k=0}^{g-1}$ is an exponential function of an arbitrary point of the Jacobian for the spectral curve of the gth kind, while the function $H^{(g)}$ is a specific limit of the Θ function for the algebraic curve degenerating into a sphere. The solution of Eqs. (169) in terms of algebraic geometry was written in [3, 11], and the reduction is described in detail in [12].

The substitution of (191) into Eqs. (180) and (181) yields

$$\begin{aligned}
x_{1,\mathbf{n}} &= q^{-1}\frac{e(X_{n_1}, Z'_{n_3})e(Y_{n_2}, Z_{n_3})}{e(X_{n_1}, Z_{n_3})e(Y_{n_2}, Z'_{n_3})}\frac{\tau_{1,\mathbf{n}}\tau_{2,\mathbf{n}+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}}},\\
y_{1,\mathbf{n}} &= \frac{e(X_{n_1}, Y_{n_2})e(Z_{n_3}, Z'_{n_3})}{e(X_{n_1}, Z_{n_3})e(Y_{n_2}, Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}}},\\
x_{2,\mathbf{n}} &= q^{-1}\frac{e(X_{n_1}, Z'_{n_3})e(Y'_{n_2}, Z_{n_3})}{e(X_{n_1}, Z_{n_3})e(Y_{n_2}, Z'_{n_3})}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}\tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}},\\
y_{2,\mathbf{n}} &= q\frac{e(X'_{n_1}, Y'_{n_2})e(Z_{n_3}, Z'_{n_3})}{e(X'_{n_1}, Z_{n_3})e(Y'_{n_2}, Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}},\\
x_{3,\mathbf{n}} &= q^{-1}\frac{e(X_{n_1}, Z'_{n_3})e(Y'_{n_2}, Z_{n_3})}{e(X_{n_1}, Z_{n_3})e(Y'_{n_2}, Z'_{n_3})}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}\tau_{2,\mathbf{n}+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}}},\\
y_{3,\mathbf{n}} &= q\frac{e(X_{n_1}, Y'_{n_2})e(Z_{n_3}, Z'_{n_3})}{e(X_{n_1}, Z_{n_3})e(Y'_{n_2}, Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}}},\\
x_{4,\mathbf{n}} &= q^{-1}\frac{e(X'_{n_1}, Z'_{n_3})e(Y_{n_2}, Z_{n_3})}{e(X'_{n_1}, Z_{n_3})e(Y_{n_2}, Z'_{n_3})}\frac{\tau_{1,\mathbf{n}}\tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}},\\
y_{4,\mathbf{n}} &= \frac{e(X'_{n_1}, Y_{n_2})e(Z_{n_3}, Z'_{n_3})}{e(X'_{n_1}, Z_{n_3})e(Y_{n_2}, Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_1}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}}.
\end{aligned}\tag{193}$$

In particular, if all $f_k \equiv 0$, then $\tau_{j,\mathbf{n}} \equiv 1$. In this case, the parametrization of each $\mathbf{R}_\mathbf{n}$ matrix is equivalent to Eqs. (167) and (168), so that Eqs. (193) describe the Zamolodchikov–Bazhanov–Baxter model in the vertex formulation and the solution of the whole linear problem is given by formulas (89)–(91).

Finally, periodic boundary conditions (182), expressed in terms of arguments of τ functions, reduce to the relations

$$I_k(\mathbf{n} + N_2\mathbf{e}_2) = I_k(\mathbf{n} + N_3\mathbf{e}_3) = I_k(\mathbf{n}). \tag{194}$$

Therefore, each pair (P_k, P'_k) should satisfy the equations

$$\begin{aligned}
\prod_{n_2=0}^{N_2-1}\frac{P'-Y'_{n_2}}{P-Y'_{n_2}} &= \prod_{n_2=0}^{N_2-1}\frac{P'-Y_{n_2}}{P-Y_{n_2}},\\
\prod_{n_3=0}^{N_3-1}\frac{P'-Z'_{n_3}}{P-Z'_{n_3}} &= \prod_{n_3=0}^{N_3-1}\frac{P'-Z_{n_3}}{P-Z_{n_3}}.
\end{aligned}\tag{195}$$

The number of nonequivalent solutions of system (195) is equal to

$$g = (N_2-1)(N_3-1). \tag{196}$$

The equivalence means that, if a pair (P, P') is a solution of Eqs. (195), then the pair (P', P) is the equivalent solution. Therefore, although the τ functions with an arbitrary number of solitons and with arbitrary parameters $P_0, \dots, P_{g-1}$ and $P'_0, \dots, P'_{g-1}$ describe the solutions locally, the periodic boundary conditions (i.e., finite volume) impose certain restrictions on the maximum number g of soliton modes and on values of the parameters $P_0, \dots, P'_{g-1}$. The quantity g in general position means the kind of classical spectral curve $J(x^N, y^N) = 0$.

5.3. T *Matrix*

It is evident that operator (183) is a matrix product of simpler objects which are associated with the n_1 layers in this expression for $\mathbf{Z}_1$ and referred to as a layer–layer transfer matrix. By analogy with two-dimensional models, we can define the following monodromy matrix: for a fixed n_1,

$$\langle\sigma_1,\sigma_2,\sigma_3|\hat{\mathbf{T}}^{(n_1)}|\sigma'_1,\sigma'_2,\sigma'_3\rangle = \sum_{\sigma_1,\sigma_2,\sigma_3}\prod_{n_1,n_2,n_3}\langle\sigma_{1:n_2,n_3},\sigma_{2:n_2,n_3},\sigma_{3:n_2,n_3}|\mathbf{R}_\mathbf{n}|\sigma'_{1:n_2,n_3},\sigma_{2:n_2+1,n_3},\sigma_{3:n_2,n_3+1}\rangle, \tag{197}$$

where the matrix elements of $\hat{\mathbf{T}}^{(n_1)}$ are evaluated for the sets

$$\begin{gathered}
\sigma_1 = \{\sigma_{1:n_2,n_3}\}, \quad \sigma_2 = \{\sigma_{2:0,n_3}\}, \quad \sigma_3 = \{\sigma_{3:n_2,0}\},\\
n_2 = 0, \dots, N_1-1, \quad n_3 = 0, \dots, N_3-1,
\end{gathered}\tag{198}$$

$$\begin{gathered}
\sigma'_1 = \{\sigma'_{1:n_2,n_3}\}, \quad \sigma'_2 = \{\sigma_{2:N_2,n_3}\}, \quad \sigma'_3 = \{\sigma_{3:n_2,N_3}\},\\
n_2 = 0, \dots, N_2-1, \quad n_3 = 0, \dots, N_3-1.
\end{gathered}\tag{199}$$

The summation is over all "inner" $\sigma_{2:n_2,n_3}$, $\sigma_{3:n_2,n_3}$, with $0 < n_2 < N_2$ and $0 < n_3 < N_3$. The layer–layer transfer matrix is the trace of the monodromy matrix. Under periodic boundary conditions (182),

$$\langle\sigma_1|\mathbf{T}^{(n_1)}|\sigma'_1\rangle = \sum_{\sigma_2,\sigma_3}\langle\sigma_1,\sigma_2,\sigma_3|\hat{\mathbf{T}}^{(n_1)}|\sigma'_1,\sigma_2,\sigma_3\rangle, \tag{200}$$

where the summation is over all $\sigma_2 = \{\sigma_{2:0,n_3} = \sigma_{2:N_2,n_3}\}$ and $\sigma_3 = \{\sigma_{3:n_2,0} = \sigma_{3:n_2,N_3}\}$. Because the problem has periodic properties, we will always assume that

$$n_2 \in \mathbb{Z}_{N_2}, \quad n_3 \in \mathbb{Z}_{N_3}. \tag{201}$$

Equation (183) can be written out in terms of $\mathbf{T}^{(n_1)}$ matrices as

$$\mathbf{Z}_1 = \prod_{n_1 = 0 \uparrow N_1 - 1} \mathbf{T}^{(n_1)}. \qquad (202)$$

In addition to **T** matrix (200), we define a degenerate **T** matrix

$$\langle \sigma_1 | \mathbf{T}^{(n_1)}(\alpha, \beta) | \sigma'_1 \rangle$$

$$= \sum_{\substack{\sigma_2 : \Sigma_{n_3} \sigma_{2:n_3} = -\beta \\ \sigma_3 : \Sigma_{n_2} \sigma_{3:n_3} = \alpha}} \langle \sigma_1, \sigma_2, \sigma_3 | \hat{\mathbf{T}}^{(n_1)} | \sigma'_1, \sigma_2, \sigma_3 \rangle \qquad (203)$$

such that

$$\mathbf{T}^{(n_1)} = \sum_{\alpha, \beta} \mathbf{T}^{(n_1)}(\alpha, \beta). \qquad (204)$$

We now consider relations similar to Eqs. (161) and (185) for each $\mathbf{T}^{(n_1)}$ matrix. For the square lattice, the functional $\mathbf{j}(x, y)$ in the first bundle depends on initial values $(u_{1:n_2, n_3}, w_{1:n_2, n_3})$, with $n_2 \in \mathbb{Z}_{N_2}$ and $n_3 \in \mathbb{Z}_{N_3}$. This is seen from formula (185). Adding dynamics by using Eq. (188), we can naturally define

$$\mathbf{j}^{(n_1)}(x, y) = \mathbf{j}(x, y | \{u_{1,\mathbf{n}}, w_{1,\mathbf{n}}\}), \qquad (205)$$

with n_1 given, $n_2 \in \mathbb{Z}_{N_2}$, $n_3 \in \mathbb{Z}_{N_3}$.

It is evident that $\mathbf{j}(x, y) \equiv \mathbf{j}^{(0)}(x, y)$ at the initial moment and the final value of functional (185) is $\mathbf{j}(x, y | \{u^*_{1:n_2, n_3}, w^*_{1:n_2, n_3}\}) \equiv \mathbf{j}^{(N_1)}(x, y)$. By definition, all $\mathbf{j}^{(N_1)}(x, y)$ are equivalent and

$$\mathbf{j}^{(n_1)}(x, y)\mathbf{T}^{(n_1)} = \mathbf{T}^{(n_1)}\mathbf{j}^{(n_1+1)}(x, y). \qquad (206)$$

It is assumed that x and y are free parameters.

We now comment on the meaning of the index n_1, transition $n_1 \mapsto n_1 + 1$, and parameters of $\mathbf{T}^{(n_1)}$ matrix entering into Eq. (206). According to parametrization (188), $u_{1,\mathbf{n}}$ and $w_{1,\mathbf{n}}$ are ratios of the functions $\tau_{2,\mathbf{n}}$ and $\tau_{3,\mathbf{n}}$ for the same n_1. For a given n_1, formula (192) can be rewritten as

$$f_k I_k(\mathbf{n}) = f_k(n_1) I_k(n_2, n_3), \qquad (207)$$

where

$$I_k(n_2, n_3) = \left(\prod_{j_2=0}^{n_2-1} \frac{\sigma_k(Y'_{j_2})}{\sigma_k(Y_{j_2})}\right)\left(\prod_{j_3=0}^{n_3-1} \frac{\sigma_k(Z'_{j_3})}{\sigma_k(Z_{j_3})}\right), \qquad (208)$$

and $f_k(n_1) = f_k \prod_{j_1=0}^{n_1-} \frac{\sigma_k(X'_{j_1})}{\sigma_k(X_{j_1})}$ is scaled amplitudes in the level n_1. In the next level, $n_1 \mapsto n_1 + 1$. The corresponding shift of all the amplitudes,

$$f_k(n_1 + 1) = f_k(n_1) \frac{\sigma_k(X'_{n_1})}{\sigma_k(X_{n_1})}, \qquad (209)$$

is parametrized by the pair (X'_{n_1}, X_{n_1}). In fact, this pair is an efficient argument of the $\mathbf{T}^{(n_1)}$ matrix, although it is rather difficult to perceive this in parametrization (193).

In the case when the soliton modes are not present, $f_k = 0$ in Eq. (192), which means that $\tau_{j,\mathbf{n}} \equiv 1$, and $\mathbf{j}^{(n_1)}(x, y) = \mathbf{j}(x, y)\ \forall n_1$. As a result, Eq. (206) transforms into commutation equation for $\mathbf{j}$ and $\mathbf{T}^{(n_1)}$. Therefore, all $\mathbf{T}^{(n_1)}$ commute. In this particular case, $\mathbf{T}^{(n_1)}$ is the transfer matrix of the nonuniform Zamolodchikov–Bazhanov–Baxter model.

5.4. Quantum Bäcklund Transformation

It is remarkable that the spectrum of the commutative operators $\mathbf{j}(x, y)$ is independent of f_k, i.e., any $\mathbf{j}^{(n_1)}(x, y)$ is isospectral to $\mathbf{j}(x, y)$ with $\tau_{j,\mathbf{n}} \equiv 1$. In order to prove this statement, we now consider a limiting case of the amplitudes f_k and X_{n_1}, X'_{n_1} entering into Eq. (191).

Let $N_1 = g \equiv (N_2 - 1)(N_3 - 1)$, so that the numbering of k and n_1 can be identified. We then renormalize the amplitude f_k,

$$f_k = f'_k \sigma_k(X_k), \qquad (210)$$

and consider the limit

$$X_k \mapsto P'_k, \quad \text{where } f'_k \text{ is arbitrary.} \qquad (211)$$

In this limit, $\sigma_k(X_k) \mapsto 0$. For an n_1 given, the arguments of the functions $\tau^N_{2,\mathbf{n}}$ and $\tau^N_{3,\mathbf{n}}$, up to regular functions $\sigma_k(Y)$ and $\sigma_k(Z)$, are

$$f_k I_k(\mathbf{n}) = f'_k \sigma_k(X_k) \prod_{j=0}^{n_1-1} \frac{\sigma_k(X'_j)}{\sigma_k(X_j)} I_k(n_2, n_3), \qquad (212)$$

where $I_k(n_2, n_3)$ is given by formula (208). In limit (211), $f_k I_k(\mathbf{n}) = 0$ for all $k \geq n_1$. According to Eqs. (191) and (192), the number of nonzero amplitudes $f_k I_k(\mathbf{n})$ in the set $(\tau_{2,\mathbf{n}}, \tau_{3,\mathbf{n}})$ is equal to n_1. This number will be referred to as the number of solitons in the set $(u_{1,\mathbf{n}}, w_{1,\mathbf{n}})$. In the particular case when $n_1 = 0$, parametrization (188) for $u_{1,\mathbf{n}}$ and $w_{1,\mathbf{n}}$ coincides with parametrization (89), i.e., $\mathbf{j}^{(0)}(x, y) = \mathbf{j}(x, y)$. In limit (211), the arguments of the function $\tau_{1,\mathbf{n}}$,

$$f_k\frac{I_k(\mathbf{n})}{\sigma_k(X_{n_1})} = f'_k\frac{\sigma_k(X_k)}{\sigma_k(X_{n_1})}\prod_{j=0}^{n_1-1}\frac{\sigma_k(X'_j)}{\sigma_k(X_j)}I_k(n_2, n_3), \quad (213)$$

involve $n_1 + 1$ nonzero soliton modes. When $k = n_1$, Eq. (213) does not degenerate. In the particular case where $n_1 = 0$, when $\tau_{2,\mathbf{n}} = \tau_{3,\mathbf{n}} = 1$, we readily derive the equation

$$\tau^N_{1,\mathbf{n}} = 1 - f'_0\prod_{j\neq 0}\frac{P'_0 - P_j}{P_0 - P_j}I_k(n_2, n_3) \quad (214)$$

from definition (189). According to [23], this is a one-soliton wave.

The $\mathbf{T}^{(n_1)}$ matrix in Eq. (206) defines a relation of n_1-soliton solutions to $(n_1 + 1)$-soliton solutions; therefore, the $\mathbf{T}^{(n_1)}$ matrix is associated with the soliton production and represents the Bäcklund transformation. In this case, the state $(u_{1,\mathbf{n}}, w_{1,\mathbf{n}})$ with $n_1 = g$ is parametrized by the complete set of soliton amplitudes. The corresponding product of $\mathbf{T}$ matrices,

$$\mathbf{K} = \prod_{k=0\uparrow g-1}\mathbf{T}^{(k)}, \quad (215)$$

is the production operator for a general soliton state.

When constructing the operator $\mathbf{K}$, we use an arbitrary ordering of the set (P_k, P'_k); therefore, the final g-soliton solution is independent of the ordering. The number of the orderings is $g!$; i.e., there exist $g!$ factorization variants of the product in Eq. (215). In all the variants, the inner $\mathbf{T}$ operators are different. Therefore, in the soliton sector, the permutation relations for the noncommutative $\mathbf{T}$ operators are rather intricate and allow for the distribution in time of soliton production events.

5.5. *Degenerate* T *Matrix and Auxiliary Linear Problem*

We now consider the relation between the $\mathbf{T}$ matrix and the auxiliary linear problem.

The degenerate $\mathbf{T}^{(n_1)}$ matrix is given by formula (203). Substituting Eq. (150) into formula (203), we remove the delta symbol by the change of variables

$$\begin{aligned}\sigma_{2:n_2,n_3} &= \zeta_{n_2-1,n_3} - \zeta_{n_2-1,n_3+1},\\ \sigma_{3:n_2,n_3} &= \zeta_{n_2,n_3} - \zeta_{n_2-1,n_3}.\end{aligned} \quad (216)$$

Because formulas (216) are difference equations, we should fix $\zeta_{0,0} \equiv 0$. By definition, the periodic conditions have the form

$$\zeta_{n_2+N_2,n_3} = \zeta_{n_2,n_3} + \alpha, \quad \zeta_{n_2,n_3+N_3} = \zeta_{n_2,n_3} + \beta. \quad (217)$$

Using substitution (216), we write the matrix elements $\mathbf{T}^{(n_1)}(\alpha, \beta)$ in the form

$$\begin{aligned}&\langle\sigma_1|\mathbf{T}^{(n_1)}(\alpha,\beta)|\sigma'_1\rangle\\ &= \sum_{\zeta}\langle\sigma_1|\phi'^{(n_1)}(\zeta)\rangle\langle\phi^{(n_1+1)}(\zeta)|\sigma'_1\rangle.\end{aligned} \quad (218)$$

Here, periodic conditions (217) and the condition $\zeta_{0,0} \equiv 0$ must be taken into account and

$$\begin{aligned}\langle\sigma_1|\phi'^{(n_1)}(\zeta)\rangle &= \prod_{n_2,n_3} q^{\sigma_{1:n_2,n_3}(\zeta_{n_2-1,n_3+1}-\zeta_{n_2,n_3+1})}\\ &\times\frac{W_{p_{1,\mathbf{n}}}(\zeta_{n_2-1,n_3}-\zeta_{n_2-1,n_3+1}-\sigma_{1:n_2,n_3})}{W_{p_{3,\mathbf{n}}}(\zeta_{n_2,n_3}-\zeta_{n_2,n_3+1}-\sigma_{1:n_2,n_3})},\\ \langle\phi^{(n_1+1)}(\zeta)|\sigma'\rangle &= \prod_{n_2,n_3} q^{\sigma'_{1:n_2,n_3}(\zeta_{n_2,n_3+1}-\zeta_{n_2-1,n_3+1})}\\ &\times\frac{W_{p_{2,\mathbf{n}}}(\zeta_{n_2,n_3}-\zeta_{n_2,n_3+1}-\sigma'_{1:n_2,n_3})}{W_{p_{4,\mathbf{n}}}(\zeta_{n_2-1,n_3}-\zeta_{n_2-1,n_3+1}-\sigma'_{1:n_2,n_3})}.\end{aligned} \quad (219)$$

The auxiliary linear problem can be solved in terms of the vectors $|\phi'^{(n_1)}(\zeta)\rangle$ and $\langle|\phi^{(n_1+1)}(\zeta)|$ in the form

$$\begin{aligned}0 &= \langle\phi^{(n_1+1)}(\zeta)|\cdot(m_{n_2,n_3} + m_{n_2,n_3+1}q^{1/2}\mathbf{u}_{1,\mathbf{n}+\mathbf{e}_1}\\ &+ m_{n_2-1,n_3}\mathbf{w}_{1,\mathbf{n}+\mathbf{e}_1} + m_{n_2-1,n_3+1}\kappa_{1,\mathbf{n}}\mathbf{u}_{1,\mathbf{n}+\mathbf{e}_1}\mathbf{w}_{1,\mathbf{n}+\mathbf{e}_1}),\end{aligned} \quad (220)$$

where

$$\begin{aligned}\mathbf{u}_{1,\mathbf{n}+\mathbf{e}_1} &= u_{1,\mathbf{n}+\mathbf{e}_1}\mathbf{x}_{1:n_2,n_3},\\ \mathbf{w}_{1,\mathbf{n}+\mathbf{e}_1} &= w_{1,\mathbf{n}+\mathbf{e}_1}\mathbf{z}_{1:n_2,n_3},\end{aligned} \quad (221)$$

$$\begin{aligned}&m_{n_2-1,n_3} = m_{n_2-1,n_3}(X'_{n_1})\\ &= q^{\zeta_{n_2-1,n_3}}\tau_{1,\mathbf{n}}\prod_{j_2=0}^{n_2-1}\xi_{2,j_2}\frac{e(X'_{n_1},Y_{j_2})}{e(X'_{n_1},Y'_{j_2})}\prod_{j_3=0}^{n_3-1}\frac{1}{\xi_{3,j_3}}\frac{e(X'_{n_1},Z_{j_3})}{e(X'_{n_1},Z'_{j_3})}.\end{aligned} \quad (222)$$

To derive Eq. (220), one should evaluate a matrix element of Eq. (220), use definition (58) and properties of the function W given by Eq. (149), and substitute Eqs. (188) and (193).

Equation (220) is representation (67) for auxiliary linear problem (63). The eigenvalues of the operators $\mathbf{m}_{S,S_0}$ for $S_0 = (0, 0)$ are determined by Eq. (222) [see Eq. (66)]. Note that the spectral parameters entering into the determinant of $\mathbf{j}^{(n_1+1)}$ are special, and their Nth powers are on a curve. In such a situation, it is convenient to introduce the notation

$$\mathbf{j}(x, y) = \mathbf{j}(X|\alpha, \beta) \Leftrightarrow x^{-1} = q^{\alpha}\prod_{n_2=0}^{N_2-1}\xi_{2,n_2}\frac{e(X, Y_{n_2})}{e(X, Y'_{n_2})},$$
$$y = q^{\beta}\prod_{n_3=0}^{N_3-1}\frac{1}{\xi_{3,n_3}}\frac{e(X, Z_{n_3})}{e(X, Z'_{n_3})}. \tag{223}$$

Equations (222) and (223) correspond to Eqs. (90) and (91), respectively. Since $\langle\phi^{(n_1+1)}|$ is a solution of the linear problem, then, according to Eq. (55),

$$\langle\phi^{(n_1+1)}(\zeta)|\cdot\mathbf{j}^{(n_1+1)}(X'_{n_1}|\alpha, \beta) = 0 \quad \forall\zeta, \tag{224}$$

where the arguments of the operator $\mathbf{j}$ are determined from cyclic properties (222) and definition (223).

Similarly, the equation for $|\phi'^{(n_1)}(\zeta)\rangle$ takes the form:

$$(m_{n_2,n_3} + m_{n_2,n_3+1}q^{1/2}\mathbf{u}_{1,\mathbf{n}} + m_{n_2-1,n_3}q\mathbf{w}_{1,\mathbf{n}}$$
$$+ m_{n_2-1,n_3+1}q\kappa_{1,\mathbf{n}}\mathbf{u}_{1,\mathbf{n}}\mathbf{w}_{1,\mathbf{n}})\cdot|\phi'^{(n_1)}(\zeta)\rangle = 0, \tag{225}$$

where

$$\mathbf{u}_{1,\mathbf{n}} = u_{1,\mathbf{n}}\mathbf{x}_{1:n_2,n_3}, \quad \mathbf{w}_{1,\mathbf{n}} = w_{1,\mathbf{n}}\mathbf{z}_{1:n_2,n_3}, \tag{226}$$

$$m_{n_2-1,n_3} = m_{n_2-1,n_3}(X_{n_1})$$
$$= q^{\zeta_{n_2-1,n_3}}\tau_{1,\mathbf{n}}\prod_{j_2=0}^{n_2-1}\xi_{2,j_2}\frac{e(X_{n_1}, Y_{j_2})}{e(X_{n_1}, Y'_{j_2})}\prod_{j_3=0}^{n_3-1}\frac{1}{\xi_{3,j_3}}\frac{e(X_{n_1}, Z_{j_3})}{e(X_{n_1}, Z'_{j_3})}. \tag{227}$$

Relation (225) describes a dual linear problem, so that all the results obtained above remain valid. Equation (225) differs from the conventional auxiliary problem by the transformation $\mathbf{z}_{1:n_2,n_3} \mapsto q\mathbf{z}_{1:n_2,n_3}$. Hence, the determinant of system (225) is given by the equation

$$\mathbf{j}'^{(n_1)}(x, y) = \mathbf{U}\mathbf{j}^{(n_1)}(x, y)\mathbf{U}^{-1}, \tag{228}$$

where [see Eq. (39)]

$$\mathbf{U} = \prod_{n_2,n_3}\mathbf{x}_{1:n_2,n_3}. \tag{229}$$

Taking Eqs. (217), (223), and (227) into account, we arrive at the equation

$$\mathbf{j}'^{(n_1)}(X_{n_1}|\alpha, \beta)\cdot|\phi'^{(n_1)}(\zeta)\rangle = 0 \quad \forall\zeta, \tag{230}$$

similar to Eq. (224). The two equations

$$\mathbf{j}'^{(n_1)}(X_{n_1}|\alpha, \beta)\mathbf{T}^{(n_1)}(\alpha, \beta)$$
$$= \mathbf{T}^{(n_1)}(\alpha, \beta)\mathbf{j}^{(n_1+1)}(X'_{n_1}|\alpha, \beta) = 0 \tag{231}$$

for the degenerate $\mathbf{T}$ matrix follow from Eqs. (224) and (230). This relation will be analyzed below.

5.6. Spectral Decomposition

Let $\mathbf{t}^{(n_1)}_{\nu_2,\nu_3}$ be commutative operators in the expansion of $\mathbf{j}^{(n_1)}(x, y)$ [see Eqs. (38) and (41)]:

$$\mathbf{j}^{(n_1)}(x, y) = \sum_{\nu_2=0}^{N_2}\sum_{\nu_3=0}^{N_3}\mathbf{t}^{(n_1)}_{\nu_2,\nu_3}(x\mathbf{W}_0)^{\nu_3}(y\mathbf{U}_0)^{\nu_2}$$
$$\equiv \sum_{m_2=0}^{N-1}\sum_{m_3=0}^{N-1}\mathbf{t}^{(n_1)}_{m_2,m_3}(x^N, y^N)(x\mathbf{W}_0)^{m_3}(y\mathbf{U}_0)^{m_2}. \tag{232}$$

Here, the operators

$$\mathbf{U}_0 = \prod_{n_3}\mathbf{x}_{1:0,n_3}, \quad \mathbf{W}_0 = \prod_{n_2}\mathbf{z}_{1:n_2,0}, \tag{233}$$

are assumed, without loss generality, to be a normalized noncommutative pair.

We choose a complete set of the eigenvectors $|\Psi^{(n_1)}_{t,\gamma}\rangle$ as

$$\mathbf{t}^{(n_1)}_{\nu_2,\nu_3}|\Psi^{(n_1)}_{t,\gamma}\rangle = |\Psi^{(n_1)}_{t,\gamma}\rangle t_{\nu_2,\nu_3},$$
$$\mathbf{W}_0|\Psi^{(n_1)}_{t,\gamma}\rangle = |\Psi^{(n_1)}_{t,\gamma}\rangle q^{\gamma}, \tag{234}$$

where the eigenvalues t_{ν_2,ν_3} of the operators $\mathbf{t}^{(n_1)}_{\nu_2,\nu_3}$ are independent of n_1 (isospectrality) and the operator $\mathbf{W}_0$ in the pair $(\mathbf{U}_0, \mathbf{W}_0)$ is taken to be diagonal. The index t of the eigenvectors stands for the set of $N_2N_3 - 1$ independent operators $\mathbf{t}_{\nu_2,\nu_3}$, and because of the normalizing condition $\mathbf{W}_0^N = 1$, the index $\gamma \in \mathbb{Z}_N$. The dual vector $\langle\Psi^{(n_1)}_{t,\gamma}|$ is defined by the orthogonality condition

$$\langle\Psi^{(n_1)}_{t,\gamma}|\Psi^{(n_1)}_{t',\gamma'}\rangle = \delta_{t,t'}\delta_{\gamma,\gamma'}. \tag{235}$$

In the eigenvector basis, the operator $\mathbf{U}_0$ is defined as

$$\mathbf{U}_0|\Psi^{(n_1)}_{t,\gamma}\rangle = |\Psi^{(n_1)}_{t,\gamma-1}\rangle, \quad \langle\Psi^{(n_1)}_{t,\gamma}|\mathbf{U}_0 = \langle\Psi^{(n_1)}_{t,\gamma+1}|, \tag{236}$$

and the operator $\mathbf{j}^{(n_1)}(x, y)$ can be expanded:

$$\mathbf{j}^{(n_1)}(x, y) = \sum_{t,\gamma,\gamma'}|\Psi^{(n_1)}_{t,\gamma}\rangle(\mathbf{j}_t(x, y))_{\gamma,\gamma'}\langle\Psi^{(n_1)}_{t,\gamma'}|, \tag{237}$$

where, according to Eq. (232),

$$(\mathbf{j}_t(x, y))_{\gamma,\gamma'} = \sum_{m_2=0}^{N-1}\delta_{\gamma+m_2,\gamma'}y^{m_2}t_{m_2}(q^{\gamma}x, y^N), \tag{238}$$

$$t_{m_2}(x, y^N) = \sum_{m_3=0}^{N-1} x^{m_3} t_{m_2, m_3}(x^N, y^N). \qquad (239)$$

In turn, permutation relation (206) implies that

$$\mathbf{T}^{(n_1)} = \sum_{t,\gamma} |\Psi_{t,\gamma}^{(n_1)}\rangle T_t \langle \Psi_{t,\gamma}^{(n_1+1)}|, \qquad (240)$$

where T_t is independent of γ because both $\mathbf{U}_0$ and $\mathbf{W}_0$ commute with $\mathbf{T}^{(n_1)}$.

As follows from the definition of the degenerate matrix $\mathbf{T}^{(n_1)}(\alpha, \beta)$,

$$\begin{aligned} \mathbf{T}^{(n_1)}(\alpha, \beta)\mathbf{U}_0 &= \mathbf{U}_0 \mathbf{T}^{(n_1)}(\alpha + 1, \beta), \\ \mathbf{T}^{(n_1)}(\alpha, \beta)\mathbf{W}_0 &= \mathbf{W}_0 \mathbf{T}^{(n_1)}(\alpha, \beta + 1). \end{aligned} \qquad (241)$$

These relations are similar to Eq. (42), provided that the pair (x, y) is related to (α, β) by Eq. (223). The following assumption, more restrictive than Eq. (206), is based on our analytical calculations performed for N_2, $N_3 = 2, 3, 4, 5$:

$$\mathbf{t}_{\nu_2, \nu_3}^{(n_1)} \mathbf{T}^{(n_1)}(\alpha, \beta) = \mathbf{T}^{(n_1)}(\alpha, \beta) \mathbf{t}_{\nu_2, \nu_3}^{(n_1+1)}. \qquad (242)$$

It follows from Eqs. (242) and (241) that

$$\begin{aligned} &\mathbf{T}^{(n_1)}(\alpha, \beta) \\ &= \sum_{t, \gamma, \gamma'} |\Psi_{t,\gamma}^{(n_1)}\rangle T_t^{(n_1)}(\alpha - \gamma, \alpha - \gamma') q^{\beta(\gamma' - \gamma)} \langle \Psi_{t,\gamma'}^{(n_1+1)}|, \end{aligned} \qquad (243)$$

and, in particular,

$$\begin{aligned} \mathbf{T}^{(n_1)}(\alpha) &= \sum_{\beta} \mathbf{T}^{(n_1)}(\alpha, \beta) \\ &= \sum_{t,\gamma} |\Psi_{t,\gamma}^{(n_1)}\rangle T_t^{(n_1)}(\alpha - \gamma) \langle \Psi_{t,\gamma}^{(n_1+1)}|. \end{aligned} \qquad (244)$$

5.7. Baxter Equation

Let us define the analytical Baxter equation as an equation for a zero eigenvector $Q_{t,\gamma}$ or for a coeigenvector $Q'_{t,\gamma}$ of the matrix $(\mathbf{j}_t(x, y))_{\gamma,\gamma'}$ given by Eq. (238):

$$\begin{aligned} &\sum_{\gamma'} (\mathbf{j}_t(x, y))_{\gamma, \gamma'} Q_{t, \gamma'} \\ &= \sum_{\gamma} Q'_{t,\gamma} (\mathbf{j}_t(x, y))_{\gamma, \gamma'} = 0. \end{aligned} \qquad (245)$$

If (x^N, y^N) is in general position of the classical spectral curve $J(x^N, y^N) = 0$, then both equations (245) have a unique, up to a common normalizing factor, solution.

Let us explain why Eqs. (247) can actually be referred to as Baxter equations. Substituting Eq. (238) into Eq. (245), we conclude that the zero eigenvectors

$$Q_{t,\gamma} = Q_t(q^{\gamma} x, y), \quad Q'_{t,\gamma} = Q'_t(q^{\gamma} x, y) \qquad (246)$$

are meromorphic functions on a quantum curve $\Gamma^Q \ni (x, y) : J(x^N, y^N) = 0$. In this case, Eqs. (245) reduce to the functional equations

$$\begin{aligned} &\sum_{m_2=0}^{N-1} y^{m_2} t_{m_2}(x, y^N) Q_t(q^{m_2} x, y) = 0, \\ &\sum_{m_2=0}^{N-1} Q'_t(q^{-m_2} x, y) y^{m_2} t_{m_2}(q^{-m_2} x, y^N) = 0. \end{aligned} \qquad (247)$$

Naturally, Q_t and Q'_t are defined up to a normalizing factor $N(x^N, y)$. When $N_2 = 2$ (and $N > 2$), the sums over ν_2 in Eq. (38) and over m_2 in all the preceding formulas have just three terms with $m_2 = 0, 1, 2$. In this case, t_0 and t_2 in Eq. (247) are trivial [similar to Eq. (40)], t_1 is equivalent to the transfer matrix for the Potts chiral model, and Eqs. (247) coincide with Baxter $(t - Q)$ equations.

However, the quantities Q and Q' taken as meromorphic functions are not convenient for our consideration. We parameterize the pairs (x, y) by the triples $(X|\alpha, \beta)$ [see Eq. (223)] and define $Q_{t,\gamma}(X)$ and $Q'_{t,\gamma}(X)$ as solutions of Eq. (245) for the pairs (x, y) corresponding to the triples $(X|0, 0)$. Then, for arbitrary α and β,

$$\begin{aligned} &\sum_{\gamma'} (\mathbf{j}_t(X|\alpha, \beta))_{\gamma,\gamma'} q^{-\gamma'\beta} Q_{t, \gamma' - \alpha}(X) \\ &= \sum_{\gamma} Q'_{t, \gamma - \alpha}(X) q^{\gamma\beta} (\mathbf{j}_t(X|\alpha, \beta))_{\gamma, \gamma'} = 0. \end{aligned} \qquad (248)$$

Equations (231) represent two Baxter equations, written in the operator form, for the two equivalent operators $\mathbf{j}^{(n_1)}$ and $\mathbf{j}^{(n_1+1)}$. Therefore, the matrix elements entering into in Eq. (243) take the form[8]

$$T_t^{(n_1)}(\alpha - \gamma, \alpha - \gamma') = \frac{Q_{t, \gamma - \alpha}(X_{n_1}) Q'_{t, \gamma' - \alpha}(X'_{n_1})}{N_t}, \qquad (249)$$

where the coefficient N_t is a normalizing factor. Unfortunately, no method of evaluating N_t is known for any scheme of calculating the coefficients Q_t. We imply at least the two evident schemes: the calculation of Q considered as a meromorphic function and the simple calculation of Q taken as a vector of algebraic comple-

[8] We take $N_2 = 0 \bmod N$, so that $\mathbf{j}'^{(n_1)} = \mathbf{j}^{(n_1)}$ in Eq. (231); otherwise, expansion (243) and (249) would be valid for $\mathbf{U}_0^{-N_2} \mathbf{T}^{(n_1)}(\alpha, \beta)$.

ments of the matrix $(\mathbf{j}_t)_{\gamma,\gamma'}$. In both schemes, the factor N_t turns out to depend on the eigenstate sector. This difficulty, namely, the significant difference between the spectrum of the Q operator and solutions of the analytical Baxter equation, has been known in the chiral Potts model.

5.8. Hypothesis on Quantum Separation of Variables

As was noted above in Section 5.4, the states $u_{1,\mathbf{n}}$ and $w_{1,\mathbf{n}}$ parametrized by soliton τ functions (191) are isospectral to the simple inhomogeneous Zamolodchikov–Bazhanov–Baxter model (i.e., to the case of unit $\tau_{2,\mathbf{n}}$ and $\tau_{3,\mathbf{n}}$). The operator $\mathbf{K}$ given by Eq. (215) implements the similarity transformation between the initial "empty" state and the final state with $g = (N_2 - 1)(N_3 - 1)$ solitons.

The hypothesis on quantum separation of variables is based on the possibility of choosing $\mathbb{C}^g$-number parameters such that eigenstates of the final operator $\mathbf{j}^{(g)}$ have the simplest structure.

We define the operator $\mathbf{K}(\vec{\alpha}, \vec{\beta})$,

$$\mathbf{K}(\vec{\alpha}, \vec{\beta}) = \mathbf{T}^{(0)}(\alpha_0, \beta_0)\mathbf{T}^{(1)}(\alpha_1, \beta_1)\ldots\mathbf{T}^{(g-1)}(\alpha_{g-1}, \beta_{g-1}), \tag{250}$$

where $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ is constructed in much the same way as operator (215), i.e., with a given ordering of (P_k, P'_k) in limit (211). As in the case of operator (215), the operator $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ defines the relation, Eq. (242), of the zero-soliton solution to the g-soliton solution. However, in addition to limit (211) in the definition of $\mathbf{K}(\vec{\alpha}, \vec{\beta})$, we now require that

$$f'_k \mapsto \prod_{j \neq k} \frac{P_k - P_j}{P'_k - P_j}. \tag{251}$$

As was noted in Section 5.4, both $\mathbf{K}$ and $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ are independent of the ordering of (P_k, P'_k).

We then consider the first operator $\mathbf{T}^{(0)}(\alpha_0, \beta_0)$ in product (250) and the auxiliary linear problem associated with it [see Eqs. (218) and (225)]. For $n_1 = 0$, the eigenvalues of the operator $\mathbf{m}_{n_2-1, n_3}(X_0 = P'_0|\alpha_0, \beta_0)$ are given by Eq. (227) and the function $\tau_{1,\mathbf{n}}$ is defined by Eq. (214). The argument of the operator $\mathbf{m}$ is parametrization (223) for a point (x, y) of a quantum curve. It follows from Eq. (251) that $m_{-1,0} = 0$, i.e., in terms of operators,

$$\mathbf{m}_{-1,0}(P'_0|\alpha_0, \beta_0)\mathbf{T}^{(0)}(\alpha_0, \beta_0) = 0. \tag{252}$$

By virtue of the natural invariance of the operator $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ with respect to the ordering of (P_k, P'_k), we can write the following relation, which is more restrictive than Eq. (252):

$$\mathbf{m}_{-1,0}(P'_k|\alpha_k, \beta_k)\mathbf{K}(\vec{\alpha}, \vec{\beta}) = 0 \quad \forall k. \tag{253}$$

This formula is related with classical separation of variables. If $\mathbf{m}^N_{n_2,n_3}$ is a classical Baker–Akhiezer function, then Eq. (253) means that the equation $\mathbf{m}^N_{-1,0}(P) = 0$ has g solutions $P = P'_k$ on the classical curve. For a quantum curve, the operator $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ is a projector onto the state on which all $\mathbf{m}_{-1,0}(P'_k|\alpha_k, \beta_k)$ have nonzero eigenvalues.

The overwhelming majority of the operators $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ are zero identically because operator (250) satisfies all equations $\mathbf{j}(P'_k|\alpha_k, \beta_k)\mathbf{K}(\vec{\alpha}, \vec{\beta}) = 0$ in addition to Eq. (253).

The structure of $\mathbf{K}(\vec{\alpha}, \vec{\beta})$ can be improved in the following way. Let $\mathcal{V}_k(\alpha, \beta)$ be a subspace of the state space such that

$$|\Phi\rangle \in \mathcal{V}_k(\alpha, \beta) \Leftrightarrow \mathbf{j}(P'_k|\alpha, \beta) = \mathbf{m}_{-1,0}(P'_k|\alpha, \beta)|\Phi\rangle = 0, \tag{254}$$

and let $\mathcal{V}_k(\alpha) = \bigcap_\beta \mathcal{V}_k(\alpha, \beta)$. It is evident that $\dim\mathcal{V}_k(\alpha, \beta) = N^{\Delta-2}$, where $\Delta = N_2N_3$, and $\dim\mathcal{V}_k(\alpha) = N^{\Delta-1}$. If $\mathcal{V}(\vec{\alpha}) = \bigcup_{k=0}^{g-1}\mathcal{V}_k(\alpha_k)$, then $\dim\mathcal{V}(\vec{\alpha}) = N^{\Delta-g}$. We denote vectors from $\mathcal{V}(\vec{\alpha})$ by $|\Phi_{\vec{\alpha},t'}\rangle$, where t' is a set of eigenvalues of $\Delta - g = N_2 + N_3 - 1$operators.

We now formulate a hypothesis on quantum separation of variables.

Assumption 1. Elements of the set t' are eigenvalues of the operator $\mathbf{W}_{n_3}$ and $\mathbf{U}_0^{-1}\mathbf{U}_{n_2}$.

Assumption 2. The operator

$$\mathbf{K}(\vec{\alpha}) = \sum_{\vec{\beta}} \mathbf{K}(\vec{\alpha}, \vec{\beta}) \tag{255}$$

can be expanded as

$$\mathbf{K}(\vec{\alpha}) = \sum_{t'} |\Phi_{\vec{\alpha},t'}\rangle\langle\xi_{t'}|. \tag{256}$$

The first assumption is based on the fact that each $\mathbf{T}^{(k)}(\alpha_k)$ and, therefore, each $\mathbf{K}(\vec{\alpha})$ commutes with all

$\mathbf{W}_{n_3}$ and $\mathbf{U}_0^{-1}\mathbf{U}_{n_2}$. In the second assumption, it is important that $|\xi_{t'}\rangle$ is independent of $\vec{\alpha}$.

The proof of the second assumption is a very laborious task and it remains unsolved so far. The reason is that limiting case (251) implies the presence of the 0/0 ambiguities in parametrization (193); hence, there are some degrees of freedom in the definition of $\langle\xi_{t'}|$. Expansion (256) has been analyzed in detail only for the quantum relativistic Toda chain at the root of unity [12]. Our next task for the future is to find an exact parametrization of the ambiguities in Eq. (193) and to evaluate $\langle\xi_{t'}|$ for the Zamolodchikov–Bazhanov–Baxter model.

On the other hand, the operator $\mathbf{K}(\vec{\alpha})$ must have the standard expansion

$$\mathbf{K}(\vec{\alpha}) = \sum_{t,\gamma}|\Psi_{t,\gamma}\rangle\left(\prod_{k=0}^{g-1}T_t^{(k)}(\alpha_k-\gamma)\right)\langle\Psi_{t,\gamma}^{(g)}| \qquad (257)$$

following from Eq. (244). The index γ belongs to the set t'. Comparing Eqs. (256) and (257), we conclude that

$$\langle\Phi_{t,\gamma}^{(g)}| \equiv \langle\xi_{t'}|, \qquad (258)$$

i.e., limit (251) for finite eigenvectors $\langle\Psi_{t,\gamma}^{(g)}|$ implies their collineation with respect to $t\backslash t'$. This collineation is not defined uniquely because the 0/0 ambiguities must be determined. However, for quantum separation of variables, we arrive at Sklyanin's formula [24, 25]

$$|\Phi_{\vec{\alpha},t'}\rangle = \sum_{t\backslash t'}|\Psi_{t,\gamma}\rangle\prod_{k=0}^{g-1}T_t^{(k)}(\alpha_k). \qquad (259)$$

6. CONCLUSIONS

In conclusion, we list some problems that have not been solved to date.

The formulation of 2 + 1-dimensional models encounters a series of difficulties. Within the framework of the local Weyl algebra as an algebra of observables, ansatz (14) and principle (130) turn out to be efficient tools for constructing exactly solvable three-dimensional models generalizing the Zamolodchikov–Bazhanov–Baxter model in vertex formulation. However, a wider class of auxiliary linear problems might exist, among them, those with a nonlocal algebra of variables.

The simplest remark concerning this point is that the parameters κ_V, being invariants in representation (130), could be defined as dynamical parameters, i.e., mapping (131) would contain one more row with κ'_1, …, κ'_3, and Eq. (132) would take a different form, but Eq. (134) would be valid as before. In this case, the evolution of $\kappa_{j,\mathbf{n}}$ on the whole lattice, coinciding with that given in [19], would be described by the four-term Hirota–Miwa equation [26], for which no quantum analogue is known. In such a case, the lattice model is similar to a model of parafermions $\mathbf{u}_{j,\mathbf{n}}$ and $\mathbf{w}_{j,\mathbf{n}}$ interacting with the classical gravitation described by $\kappa_{j,\mathbf{n}}$.

Most of the problems concerning the Zamolodchikov–Bazhanov–Baxter model have not been solved conclusively to date. As was noted, one of the basic problems is to find spectra of quantum transfer matrices (249) with normalizing factors depending on the sector t. However, the quantity Q when evaluated by formula (249) as a vector of algebraic complements to $(\mathbf{j}_t)_{\gamma,\gamma}$ [e.g., in representation (85) and in thermodynamic limit (105) and (111)] is close to the correct expression found in [14]. The quantities β_k in parametrization (106) are just linear Baxter excesses. In this case, the normalizing factor is an exponent of logarithms of sines of the linear excesses, while Eq. (111) contains all dilogarithms needed.

Of course, the assumption on quantum separation of variables in the Zamolodchikov–Bazhanov–Baxter model has not been proved yet. It is necessary to explicitly find a method of resolving uncertainties in the parametrization of the operator $\mathbf{K}(\vec{\alpha})$ and then evaluate $\langle\xi'|$ within the framework of this method. This task involves the technique of the expansion of function (189) in the neighborhoods of its zeros.

Then, infinite-dimensional representations of an algebra of observables should also be mentioned. The scheme presented in this paper can be dualized modularly, provided that $q = \exp(i\pi\tau)$ and $\tau = \exp 2(i\theta)$, so all the operators become unitary. For appropriate auxiliary lattices and evolutionary mappings, our scheme can be applied to the quantum Liouville model [29] and its various generalizations (e.g., two interacting Liouville fields), as well as to strongly coupled the quantum relativistic Toda chain [28]. The auxiliary lattice corresponding to the Toda chain was mentioned above in the examples. Thus, for an arbitrary evolutionary lattice, we could speak on a new three-dimensional quantum field theory.

In our opinion, a simple, but very important, conclusion is that all the commutative operators $\mathbf{t}_{\mu,\nu}$ for a finite-dimensional algebra of observables with $N = 2$ are Hermitian. This indicates that the evolutionary models have a physical sense. Moreover, in the case of a nondegenerate classical spectral curve of a nonzero kind, the brief analysis of the thermodynamic limit presented above suggests that these models have a nontrivial phase structure. (Recall that the Zamolodchikov–Bazhanov–Baxter is critical unless one of the lattice dimensions is finite; this case is equivalent to the generalized chiral Potts model.)

In this paper, we dealt only with classical spectral curves of the first kind. We need not have used the corresponding parametrization of quantum $\mathbf{T}$ matrices or statistical sum Z, although such a parametrization takes a simple form in the notations introduced above. Namely, one should set in Eqs. (186)–(188), (191), and (193) $e(X, Y)^N = \theta_1(X - Y)$, $\tau^N_{1,\mathbf{n}} = \theta_1(f - X_{n_1} + I(\mathbf{n}))$, $\tau^N_{2,\mathbf{n}} = \theta_1(f - Y_{n_2} + I(\mathbf{n}))$, $\tau^N_{3,\mathbf{n}} = \theta_1(f - Z_{n_3} + I(\mathbf{n}))$, and $\lambda^N_{\mathbf{n}} = \theta_1(f + Z'_{n_3} - X_{n_1} - Y_{n_2} + I(\mathbf{n}))$, where $I(\mathbf{n}) = \sum_{j_1=0}^{n_1-1} X'_{j_1} - X_{j_1} + \sum_{j_2=0}^{n_2-1} Y'_{j_2} - Y_{j_2} + \sum_{j_3=0}^{n_3-1} Z'_{j_3} - Z_{j_3}$ and $\theta_1(z) = \theta_1(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1/2)^2 + 2i\pi(z+1/2)(n+1/2)}$. Strictly speaking, this parametrization refers to the case when the classical spectral curve degenerates into a curve of the first kind without singularities. However, this parametrization becomes complete if the statistical sum Z is defined on a sublattice with a period of two: $X^{\#}_{n_1} = X^{\#}_{n_1 \bmod 2}$, $Y^{\#}_{n_2} = Y^{\#}_{n_2 \bmod 2}$, $Z^{\#}_{n_3} = Z^{\#}_{n_3 \bmod 2}$, and, without loss of generality, $X'_0 - X_0 + X'_1 - X_1 = Y'_0 - Y_0 + Y'_1 - Y_1 = Z'_0 - Z_0 + Z'_1 - Z_1 = \pi$. This is precisely the parametrization that corresponds to the checkerboard lattice considered in Section 3.4.2, while the simplified checkerboard lattice requires that additional restrictions $X'_j = X_j + \pi/2$, etc., be imposed.

The statistical model complying with a simplified elliptic parametrization was first proposed in [30] as the simplest integrable model corresponding to the so-called modified tetrahedron equation (if the vertex formulation is taken as coinciding with the “IRC” formulation). The vertex formulation of the modified tetrahedron equation was considered in [31]. A universal approach to the modified tetrahedron equation and to the tetrahedron equation with complex weights was formulated in [32] and [13]. In this approach, invariants of the tetrahedron equation are algebraic curves and spectral arguments of the weights are meromorphic functions on the curves. However, no one has attempted to evaluate the statistical sum for these models on a cubic lattice (even for the Mangazeev–Stroganov model) or study their spectra. In our opinion, such an analysis is one of the important problems to be solved.

ACKNOWLEDGMENTS

I am grateful to S. Pakuliak, G. von Gehlen, A. Isaev, V. Bazhanov, V. Mangazeev, R. Kashaev, Yu. Stroganov, G. Pron’ko, I. Korepanov, and J.-M. Maillard. These persons are the coauthors of the papers reviewed here. I also thank to V. Mangazeev, M. Batchel, and T. Merrel for their support and hospitality during my visit to the Australian National University.

This work was supported by the US Civilian Research and Development Foundation for the Independent States of the Former Soviet Union (grant no. RM1-2334-MO-02), the INTAS (grant no. 00-00055), the Russian Foundation for Basic Research (project no. 01-01-00201), and Australian Research Council.

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