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EVOLUTION OPERATOR FOR A QUANTUM PENDULUM

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We conjecture one remarkable relation for infinite series in the simple Weyl algebra. This relation expresses an evolution operator for a quantum pendulum via its Hamiltonian.

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The compact q-dilogarithm (or q-exponent) is by definition [1]

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-q^n x) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}x^n, \tag{1}$$

where $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$ and it is assumed that $|q|<1$. If the arguments of q-dilogarithms belong to the simple Weyl algebra

$$\mathbf{u}\mathbf{v} = q\mathbf{v}\mathbf{u}, \tag{2}$$

the q-dilogarithms are used to construct the group of rational canonical transformations of (2). For example, the operator-valued function $(\mathbf{u};q)_\infty$ implements the transformation

$$(\mathbf{u};q)_\infty \mathbf{v}(\mathbf{u};q)_\infty^{-1} = \mathbf{v}(1-\mathbf{u})^{-1}. \tag{3}$$

Several remarkable relations for the q-series over simple Weyl algebra (2) can be applied to quantum mechanics [1]. The addition theorem and the pentagon identity are among the best known:

$$(\mathbf{u};q)_\infty(\mathbf{v};q)_\infty = (\mathbf{u}+\mathbf{v};q)_\infty, \qquad (\mathbf{v};q)_\infty(\mathbf{u};q)_\infty = (\mathbf{u}-\mathbf{v}\mathbf{u}+\mathbf{v};q)_\infty. \tag{4}$$

Here we present a more complicated result concerning the “five-term function”

$$\mathbf{s} = (a_1\mathbf{v}^{-1};q)_\infty(b_0\mathbf{u};q)_\infty(-c_0\mathbf{v}\mathbf{u};q)_\infty(a_0\mathbf{v};q)_\infty(b_1\mathbf{u}^{-1};q)_\infty, \tag{5}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the Weyl generators and $a_0, a_1, b_0, b_1, c_0 \in \mathbb{C}$. Let

$$\mathbf{h} = H(\mathbf{u},\mathbf{v}) = a_0\mathbf{v} + b_0\mathbf{u} + a_1\mathbf{v}^{-1} + b_1\mathbf{u}^{-1} - c_0\mathbf{v}\mathbf{u} - c_1\mathbf{v}^{-1}\mathbf{u}^{-1}, \tag{6}$$

where $c_1 = a_1 b_1$. The polynomial $\mathbf{h}$ is the Hamiltonian of $\mathbf{s}$ in the following sense. The operator $\mathbf{s}$ defines the rational transformations

$$\mathbf{u} \mapsto \mathbf{u}' = \mathbf{s}\mathbf{u}\mathbf{s}^{-1}, \qquad \mathbf{v} \mapsto \mathbf{v}' = \mathbf{s}\mathbf{v}\mathbf{s}^{-1}, \tag{7}$$

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where, for example, $\mathbf{u}'$ can be obtained by repeated use of (3):

$$\mathbf{u}' = \tilde{\mathbf{u}} - \mathbf{v}(a_0 - c_0\tilde{\mathbf{u}})\tilde{\mathbf{u}}(1 - b_0\tilde{\mathbf{u}})^{-1}, \qquad \tilde{\mathbf{u}} = \mathbf{u}(1 - a_1\mathbf{v}^{-1})^{-1}. \tag{8}$$

It can be verified "by hand" that $H(\mathbf{u}', \mathbf{v}') = H(\mathbf{u}, \mathbf{v})$, i.e., $\mathbf{h}$ is indeed an invariant of $\mathbf{s}$,

$$\mathbf{sh} = \mathbf{hs}. \tag{9}$$

Polynomial (6) is sometimes called the Hamiltonian for a kind of quantum pendulum, although its form and the form of the evolution mapping is rather unusual.

The operator $\mathbf{s}$ and its invariant $\mathbf{h}$ appeared in [2] as the respective evolution operator and the Hamiltonian of a simplest case of a hierarchy of (2+1)-dimensional integrable systems. The following conjecture was formulated in [2].

Conjecture 1. *The operator-valued function* $\mathbf{s}$*, defined by* (5) *as the series in* $\mathbf{u}$ *and* $\mathbf{v}$

$$\mathbf{s} = \sum_{n,m\in\mathbb{Z}} f_{n,m}(a_0, a_1, b_0, b_1, c_0)\mathbf{v}^n\mathbf{u}^m, \tag{10}$$

can be rewritten identically as the $\mathbf{v}^n\mathbf{u}^m$ *expansion of the series*

$$\mathbf{s} = \sum_{k=0}^{\infty} g_k(a_0a_1, b_0b_1, c_0a_1b_1)\mathbf{h}^k \tag{11}$$

for generic a_0*,* a_1*,* b_0*,* b_1*, and* c_0*.*

To obtain (10) from (11), we must do an *infinite* partial resummation. Let

$$\mathbf{h}^k = \sum_{n,m=-k}^{n,m=k} \chi_{k,n,m}\mathbf{v}^n\mathbf{u}^m. \tag{12}$$

Then the statement of Conjecture 1 is that

$$f_{n,m} = \sum_{k=0}^{\infty} g_k\chi_{k,n,m} \tag{13}$$

are convergent series. We note that identities (4), being particular cases of the representation of $\mathbf{s}$ as a function of the corresponding $\mathbf{h}$, contain *finite* resummations like (13).

Next, we can guess the form of function (11). For brevity, we set

$$a = a_0a_1, \qquad b = b_0b_1, \qquad c = c_0a_1b_1. \tag{14}$$

For arbitrary complex h, a, b, and c, let

$$S(x) = \begin{pmatrix} 1 - hx + bx^2 & 1 \\ q^{-1}x(1-x)(a - q^{-1}cx) & 0 \end{pmatrix} \tag{15}$$

and

$$S_n(x) = S(x) \cdot S(qx) \cdot S(q^2x) \cdots S(q^{n-1}x). \tag{16}$$

For $|q| < 1$, we easily find that

$$S_\infty(x) = \begin{pmatrix} s(x) & s(x) \\ \tilde{s}(x) & \tilde{s}(x) \end{pmatrix}, \qquad \tilde{s}(x) = q^{-1}x(1-x)(a - q^{-1}cx)s(qx). \tag{17}$$

By definition, $s(x)$ is also a function of a, b, c, and h:

$$s(x) = s(a,b,c;h;x), \qquad s(a,b,c;h;1) = s(b,a,c;h;1). \tag{18}$$

The second equality is the remarkable symmetry well hidden in definition (17) of s. In an equivalent formulation, if $F(x)$ is the solution of the functional equation

$$F(x) = 1 - hx + bx^2 + \frac{x(1-qx)(a-cx)}{F(qx)} \tag{19}$$

with $F(0) = 1$, then $s(x) = \prod_{n=0}^{\infty} F(q^n x)$.

Conjecture 2. *As a function of a, b, c, and $\mathbf{h}$, the operator $\mathbf{s}$ is given by*

$$\mathbf{s} = s(a,b,c;\mathbf{h};1). \tag{20}$$

For $\mathbf{s}$ given by (20), we have verified the conjectured identities (13) analytically as the series in q up to q^{21}.

The origin of the conjectured formula (20) for $\mathbf{s}$ can be clarified as follows.

Classical limit. We first consider the situation where $q = 1$; then $\mathbf{u}, \mathbf{v}, \mathbf{h} = u, v, h \in \mathbb{C}$, where $h = H(u,v)$ is given by (6) with commutative u and v. It is well known [3] that instead of Weyl algebra relation (2), we then have the Poisson algebra $\{u, v\} = uv$ at the classical level.

The equations of motion (e.g., for u) are then

$$\frac{du}{dt} \equiv \dot{u} = \{h, u\}, \tag{21}$$

and we can eliminate v and rewrite them as

$$\dot{u}^2 = u^2(h - b_0 u - b_1 u^{-1})^2 - 4u^2(a_0 - c_0 u)(a_1 - c_1 u^{-1}). \tag{22}$$

Of course, the right-hand side of (22) is the discriminant of $H(u,v) = h$ with respect to v. The function v in terms of u and $\dot{u}$ is then

$$v = \frac{h - b_0 u - b_1 u^{-1} - \dot{u}u^{-1}}{2(a_0 - c_0 u)}. \tag{23}$$

The transformation $u, v \mapsto u', v'$ given by (7) is equivalent to a shift[1] $t \mapsto t + \tau$, where

$$\tau = \int_u^{u'} \frac{du}{\dot{u}}. \tag{24}$$

The integration limits are actually two points on the elliptic curve $H(u, v) = h$. Clearly, τ is independent of the initial and terminal points on the curve. A remarkably simple choice of the initial point is $u = 0$, and because the product uv is finite, we have $\dot{u} = u' = b_1$. Upon rescaling $u = b_1 x$, this gives

$$\tau(h) = \int_0^1 \frac{dx}{\sqrt{(1 - hx + bx^2)^2 + 4x(1 - x)(a - cx)}}. \tag{25}$$

We define the "action"

$$\mathcal{A}(h) = \int^h d\chi\, \tau(\chi). \tag{26}$$

Evidently, treating this quantity as the action, we regard $\int du/\dot{u}$ as the angle. The lower limit in (26) is inessential. Integrating over χ, we obtain

$$\mathcal{A} = \mathcal{A}(a, b, c; h) = \mathcal{A}(b, a, c; h) =$$
$$= -\int_0^1 \frac{dx}{x} \log\left(\frac{1 - hx + bx^2 + \sqrt{(1 - hx + bx^2)^2 + 4x(1 - x)(a - cx)}}{2}\right). \tag{27}$$

To establish the symmetry $a \leftrightarrow b$, it is convenient to reparameterize the curve $H(u, v) = h$ as follows. Let

$$u = b_1 x, \qquad v = -\frac{(1 - x)}{x(a_0 - c_0 b_1 x)} y^{-1}. \tag{28}$$

Then

$$h = x^{-1} + y^{-1} - x^{-1} y^{-1} + bx + ay - cxy. \tag{29}$$

Formula (27) can then be rewritten in the manifestly symmetric form

$$\mathcal{A}(h) = \frac{\pi^2}{6} + \int_{\text{in}}^{\text{out}} \log y \, d \log x, \tag{30}$$

where in $= (x = 0, y = 1)$, out $= (x = 1, y = 0)$, and x and y are related by (29). The extra constant

$$\frac{\pi^2}{6} = -\int_0^1 \frac{dx}{x} \log(1 - x).$$

We note that integrals of type (30) appeared in [4].

[1] The solution of the equations of motion is as follows. We first define $\sigma(x) = \theta_1(x)/\theta_1(x + \eta)$. Then in the case of general position, we have

$$v = v(t) = \frac{\sigma(t - \alpha_2)}{\sigma(t - \alpha_3)}, \qquad u = u(t) = \frac{\sigma(t - \alpha_1)}{\sigma(t - \alpha_2)},$$

$$a_0 = -\frac{\sigma(\alpha_2 - \alpha_3)\sigma(\alpha_1 - \alpha_2)}{\sigma(\alpha_3 - \alpha_2)\sigma(\alpha_1 - \alpha_3)}, \qquad a_1 = \frac{\sigma(\alpha_1 - \alpha_2)}{\sigma(\alpha_1 - \alpha_3)},$$

$$b_0 = -\frac{\sigma(\alpha_2 - \alpha_3)\sigma(\alpha_1 - \alpha_2)}{\sigma(\alpha_2 - \alpha_1)\sigma(\alpha_1 - \alpha_3)}, \qquad b_1 = \frac{\sigma(\alpha_2 - \alpha_3)}{\sigma(\alpha_1 - \alpha_3)},$$

$$c_0 = -\frac{\sigma(\alpha_2 - \alpha_3)\sigma(\alpha_1 - \alpha_2)}{\sigma(\alpha_3 - \alpha_1)\sigma(\alpha_1 - \alpha_3)}, \qquad c_1 = \frac{\sigma(\alpha_2 - \alpha_3)\sigma(\alpha_1 - \alpha_2)}{\sigma(\alpha_1 - \alpha_3)^2}.$$

The variable t is here proportional to the variable used in (22). The periodicity of this solution explains the term "pendulum." The time shift τ is proportional to $\alpha_1 - \alpha_3 + \eta$.

Quantization. We now turn to the quantum case of Weyl algebra (2). We formally consider the representation of (2) as

$$\mathbf{u}|x\rangle = |x\rangle x b_1, \qquad \mathbf{v}|x\rangle = |qx\rangle \frac{-1}{x(a_0 - c_0 b_1 x)}. \tag{31}$$

With this definition, it can be verified that

$$\mathbf{u}|0\rangle = 0, \qquad \mathbf{vu}|0\rangle = |0\rangle\left(-\frac{b_1}{a_0}\right), \tag{32}$$

and using explicit form (8), we obtain

$$\mathbf{u}'|0\rangle = |0\rangle b_1. \tag{33}$$

These relations are the analogues of (28). Because $\mathbf{u}' = \mathbf{sus}^{-1}$, Eq. (33) implies $\mathbf{s}^{-1}|0\rangle = |1\rangle$. If $\langle\psi_h|$ is defined as $\langle\psi_h|\mathbf{s} = s(a,b,c;h)\langle\psi_h|$, then

$$s(a,b,c;h) = \frac{\langle\psi_h|0\rangle}{\langle\psi_h|1\rangle}.$$

This expression for s in terms of $\langle\psi_h|x\rangle$ coincides with (20) because the "Schödinger equation" for $\langle\psi_h|x\rangle \equiv \psi_h(x)$ is

$$\langle\psi_h|(\mathbf{h}-h)|x\rangle = \psi_h(x)(1-h+bx^2) - \psi_h(qx) + \psi_h(q^{-1}x)q^{-1}x(1-x)(a-cq^{-1}x) = 0. \tag{34}$$

Matrix (15) appears in rewriting (34) as the matrix one-step recursion. On the other hand, (20) is the appropriate discretization (i.e., the quantization) of formula (27). In particular, we can establish the assymptotic behavior

$$s(a,b,c;h;1) = \exp\bigl\{-\epsilon^{-1}\mathcal{A}(a,b,c;h) + O(\epsilon^0)\bigr\} \tag{35}$$

as $q = e^{-\epsilon}$, $\epsilon \to 0$.

Our way to "derive" $\mathbf{s} = s(\mathbf{h})$ in terms of the wave functions is rather formal, of course. Our considerations become meaningful when considering the pair of Shrödinger equations within the modular double construction [5]. A result of our considerations is that the modular dual evolution operator $\mathbf{s}$, defined as the product of five noncompact dilogarithms in the same way as in (5), is equal to the noncompact version of the function $s_{q,\tilde{q}}(a,b,c;h)$, which is the ratio of two s-functions (18) defined for $q = e^{2\pi i\tau}$ and for its modular dual $\tilde{q} = e^{-2\pi i/\tau}$.

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