

# Quantum geometry of 3-dimensional lattices

Vladimir V. Bazhanov, Vladimir V. Mangazeev and Sergey M. Sergeev

*Department of Theoretical Physics,  
Research School of Physical Sciences and Engineering,  
Australian National University, Canberra, ACT 0200, Australia.*

## Abstract

We study geometric consistency relations between angles on 3-dimensional (3D) circular quadrilateral lattices — lattices whose faces are planar quadrilaterals inscribable into a circle. We show that these relations generate canonical transformations of a remarkable “ultra-local” Poisson bracket algebra defined on discrete 2D surfaces consisting of circular quadrilaterals. Quantization of this structure leads to new solutions of the tetrahedron equation (the 3D analog of the Yang-Baxter equation). These solutions generate an infinite number of non-trivial solutions of the Yang-Baxter equation and also define integrable 3D models of statistical mechanics and quantum field theory. The latter can be thought of as describing quantum fluctuations of lattice geometry. The classical geometry of the 3D circular lattices arises as a stationary configuration giving the leading contribution to the partition function in the quasi-classical limit.

# 1 Introduction

Currently the quantum integrability is mostly understood as a *purely algebraic* phenomenon. It stems from the Yang-Baxter equation [1, 2] and other algebraic structures such as the affine quantum groups [3, 4] (also called the quantized Kac-Moody algebras), the Virasoro algebra [5] and their representation theory. It is, therefore, quite interesting to learn that some integrable models of statistical mechanics and quantum field theory arise also in quantization of classical discrete geometries. Recently, we have shown [6, 7] that the two-dimensional integrable model associated with the Faddeev-Volkov solution [8–10] of the Yang-Baxter equation describes quantum fluctuations of circle patterns [11] connected with the Thurston’s discrete analogues of the conformal transformations of the two-dimensional plane [12]. The Faddeev-Volkov model is an Ising-type model with continuous spin variables, which are interpreted as fluctuating radii of the circles. It contains a free parameter, which can be identified with the Planck constant (in the language of Euclidean quantum field theory) or with the temperature (in the language of statistical mechanics). The classical geometry of the circle patterns is described by stationary configurations giving the leading contribution to the partition function in the quasi-classical (or the zero-temperature) limit.

In this paper we consider similar connections between integrable *three-dimensional* (3D) quantum systems and integrable classical models of 3D discrete geometry. The analog of the Yang-Baxter equation for integrable quantum systems in 3D is called the *tetrahedron equation*. It was introduced by Zamolodchikov in [13, 14] (see also [15–21] for further results in this field, used in this paper). Similarly to the Yang-Baxter equation the tetrahedron equation provides local integrability conditions which are not related to the size of the lattice. Therefore the same solution of the tetrahedron equation defines different integrable models on lattices of different size, e.g., for finite periodic cubic lattices. Obviously, any such three-dimensional model can be viewed as a two-dimensional integrable model on a square lattice, where the additional third dimension is treated as an internal degree of freedom. Therefore every solution of the tetrahedron equation provides an infinite sequence of integrable 2D models differing by the size of this “hidden third dimension”. Then a natural question arises whether known 2D integrable models can be obtained in this way. A complete answer to this question is yet unknown. So far only two different (but related) examples of such correspondence have been constructed. The first example, connected with the generalized chiral Potts model [22, 23], was found in [16]. The second example was recently found in [21]. The corresponding solution of the tetrahedron equation constructed in [21] reproduces all two-dimensional solvable models related to finite-dimensional highest weight representations for all quantized affine algebras  $U_q(\widehat{sl}_n)$ ,  $n = 2, 3, \dots, \infty$ , where the rank  $n$  coincides with the size of the hidden dimension. Here we unravel yet another remarkable property of the same solution of the tetrahedron equation. We show that it can be obtained from quantization of geometric integrability conditions for the 3D *quadrilateral lattices*, i.e., lattices constructed from planar quadrilaterals [24, 25].

The classical integrability of the quadrilateral lattices was discovered in [25]. These lattices were introduced there as discretizations of triply orthogonal coordinate systems, originating

from the works of Lamé [26] and Darboux [27] on the *isothermic surfaces* — surfaces of constant temperature within solids. Such coordinate systems are described by partial differential equations, integrable by the methods of the classical soliton theory [28]. The literature on the discrete isothermic surfaces, the quadrilateral lattices and the associated integrable difference equation is very large (see, e.g., [29–32] and references therein). Nonetheless, to our knowledge the discrete integrable equations, considered in this paper have not been hitherto studied in this context. The integrability conditions for these equation are presented here in the standard form of the functional tetrahedron equation [20] (the classical analog of the quantum tetrahedron equation, discussed above). Moreover we show that these algebraic integrability conditions have a very transparent geometric meaning: they merely reflect the fact of existence of the 4D Euclidean cube.

The key idea of the geometric approach [24, 29–31] to the integrability of the discrete classical systems is to utilize various consistency conditions arising from geometric relation between elements of the lattice. Here we consider relations between edge angles on the 3D circular quadrilateral lattices — the lattices whose faces are planar quadrilaterals inscribable into a circle. We show these relations describe symplectic transformations of a remarkable “ultra-local” Poisson algebra on quadrilateral surfaces (see Eq.(25)). In Section 3 we formulate a variational principle which generates these angle relations and explicitly calculate the Lagrangian form of the action functional  $\mathcal{S}^{(cl)}$  (curiously, it expressed through the Lobachevsky function). Next, in Section 4 we quantize this structure and obtain two solutions of the tetrahedron equation. One of them was previously known [21], but another one, given in Section 4.3 and 4.4, is new. The solutions are used to define two different integrable models of statistical mechanics and quantum field theory on the cubic lattice. Their partition functions depend on the quantum parameter  $\hbar$  (the Planck constant). For both of these models the quasi-classical limit of the partition function

$$Z \simeq \exp\left(-\frac{\mathcal{S}^{(cl)}}{\hbar}\right), \quad \hbar \rightarrow 0, \quad (1)$$

is governed by the above classical action  $\mathcal{S}^{(cl)}$ , determined the angle geometry of the circular lattices (more precisely, the two models lead to two different regimes of the classical action connected by an analytic continuation). In this paper we only state our main results, the details will be presented elsewhere [33].

## 2 Classical discrete geometry: “Existence as integrability”

In this section we consider classical discrete integrable systems associated with the quadrilateral lattices. There are several ways to extract algebraic integrable systems from the geometry of these lattices. One approach, developed in [24, 29–31], leads to discrete analogs of the Kadomtsev-Petviashvili integrable hierarchy. Here we present a different approach exploiting the angle geometry of the 3D quadrilateral lattices.

### 2.1 Quadrilateral lattices

Consider three-dimensional lattices, obtained by embeddings of the integer cubic lattice  $\mathbb{Z}^3$  into the  $N$ -dimensional Euclidean space  $\mathbb{R}^N$ , with  $N \geq 3$ . Let  $\mathbf{x}(m) \in \mathbb{R}^N$ , denote coordinates of the lattice vertices, labeled by the 3-dimensional integer vector  $m = m_1e_1 + m_2e_2 + m_3e_3 \in \mathbb{Z}^3$ , where  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$  and  $e_3 = (0, 0, 1)$ . Further, for any given lattice vertex  $\mathbf{x}_0 = \mathbf{x}(m)$ , the symbols  $\mathbf{x}_i = \mathbf{x}(m + e_i)$ ,  $\mathbf{x}_{ij} = \mathbf{x}(m + e_i + e_j)$ , etc., will denote neighboring lattice vertices.

The lattice is called *quadrilateral* if all its faces  $(\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{ij})$  are planar quadrilaterals. The existence of these lattices is based on the following elementary geometry fact (see Fig. 1) [29],

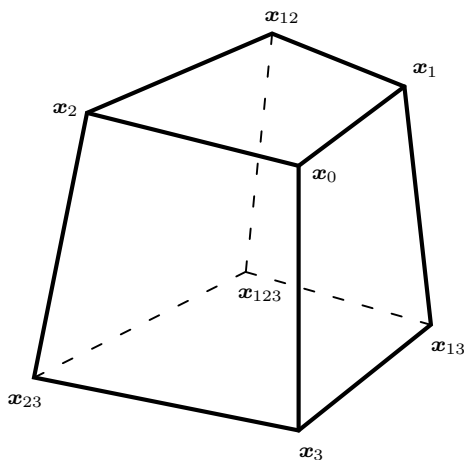


Figure 1: An elementary hexahedron of a cubic quadrilateral lattice.

Consider four points  $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  in general position in  $\mathbb{R}^N$ ,  $N \geq 3$ . On each of the three planes  $(\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j)$ ,  $0 \leq i < j \leq 3$  choose an extra point  $\mathbf{x}_{ij}$  not lying on the lines  $(\mathbf{x}_0, \mathbf{x}_i)$ ,  $(\mathbf{x}_0, \mathbf{x}_j)$  and  $(\mathbf{x}_i, \mathbf{x}_j)$ . Then there exist a unique point  $\mathbf{x}_{123}$  which simultaneously belongs to the three planes  $(\mathbf{x}_1, \mathbf{x}_{12}, \mathbf{x}_{13})$ ,  $(\mathbf{x}_2, \mathbf{x}_{12}, \mathbf{x}_{23})$  and  $(\mathbf{x}_3, \mathbf{x}_{13}, \mathbf{x}_{23})$ .

The six planes, referred to above, obviously lie in the same 3D subspace of the target space. They define a hexahedron with quadrilateral faces, shown in Fig. 1. It has the topology of the cube, so we will call it “cube”, for brevity. Let us study elementary geometry relations among the angles of this cube. Denote the angles between the edges as in Fig. 2. Altogether we have

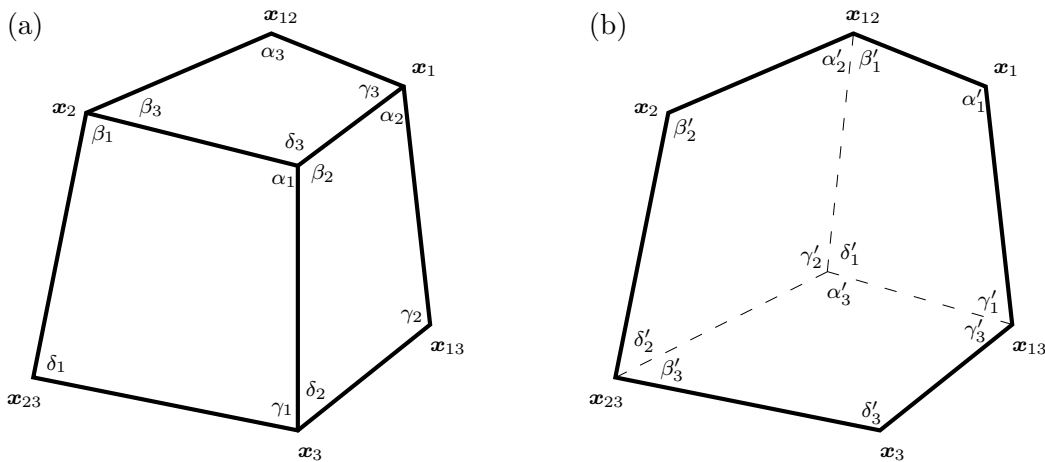


Figure 2: The “front” (a) and “back” (b) faces of the cube in Fig. 1 and their angles.

$6 \times 4 = 24$  angles, connected by six linear relations

$$\alpha_j + \beta_j + \gamma_j + \delta_j = 2\pi, \quad \alpha'_j + \beta'_j + \gamma'_j + \delta'_j = 2\pi, \quad j = 1, 2, 3, \quad (2)$$

which can be immediately solved for all “ $\delta$ ’s”. This leaves 18 angles, but only nine of them are independent. Indeed, a mutual orientation of 6 planes in the 3D-space is determined by

nine angles only. Once this orientation is fixed all other angles can be calculated. Thus the nine independent angles of the three “front” faces of the cube, shown in Fig.2a, completely determine the angles on the three “back” faces, shown in Fig.2b, and vice versa. So the geometry of our cube provides an invertible map for three triples of independent variables

$$\mathcal{R}_{123} : \quad \{\alpha_j, \beta_j, \gamma_j\} \rightarrow \{\alpha'_j, \beta'_j, \gamma'_j\}, \quad j = 1, 2, 3. \quad (3)$$

Suppose now that all angles are known. To completely define the cube one also needs to specify lengths of its three edges. All the remaining edges can be then determined from simple linear relations. Indeed, the four sides of every quadrilateral are constrained by two relations, which can be conveniently presented in the matrix form

$$\begin{pmatrix} \ell'_p \\ \ell'_q \end{pmatrix} = X \begin{pmatrix} \ell_p \\ \ell_q \end{pmatrix}, \quad X = \begin{pmatrix} A(\mathcal{A}) & B(\mathcal{A}) \\ C(\mathcal{A}) & D(\mathcal{A}) \end{pmatrix} = \begin{pmatrix} \frac{\sin \gamma}{\sin \delta} & \frac{\sin(\delta+\beta)}{\sin \delta} \\ \frac{\sin(\delta+\gamma)}{\sin \delta} & \frac{\sin \beta}{\sin \delta} \end{pmatrix} \quad (4)$$

where  $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$  denotes the set of angles and  $\ell_p, \ell_q, \ell'_p, \ell'_q$  denote the edge lengths, arranged as in Fig.3. Note that due to (2) the entries of the two by two matrix in (4) satisfy the relation

$$AD - BC = (AB - CD)/(DB - AC). \quad (5)$$

Assume that the lengths  $\ell_p, \ell_q, \ell_r$ , on one side of the two pictures in Fig.4 are given. Let us find the other three lengths  $\ell'_p, \ell'_q, \ell'_r$  on their opposite side, by iterating the relation (4). Obviously, this can be done in two different ways: either using the front three faces, or the back ones — the results must be the same. This is exactly where the geometry gets into play. The results must be consistent due to the very existence of the cube in Fig. 1 as a geometric body. However, they will be consistent only if all geometric relations between the two set of angles

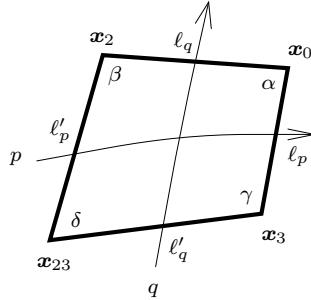


Figure 3: The angles  $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$  and sides  $\ell_p, \ell_q, \ell'_p, \ell'_q$  of a quadrilateral and the rapidity lines.

in the front and back faces of the cube are taken into account. To write these relations in a convenient form we need to introduce additional notations. Note, that Fig.3 shows two thin lines, labeled by the symbols “ $p$ ” and “ $q$ ”. Each line crosses a pairs of opposite edges, which we call “corresponding” (in the sense that they correspond to the same thin line). Eq.(4) relates the lengths  $(\ell_p, \ell_q)$  of two adjacent edges with the corresponding lengths  $(\ell'_p, \ell'_q)$  on the opposite side of the quadrilateral.

Consider now Fig.4a which contains three thin lines connecting corresponding edges of the three quadrilateral faces. By the analogy with the 2D Yang-Baxter equation, where similar arrangements occur, we call them “rapidity” lines<sup>1</sup>. We will now apply (4) three times starting

<sup>1</sup>However, at the moment we do not assume any further meaning for these lines apart from using them as a convenient way of labeling to the corresponding (opposite) edges of quadrilaterals.

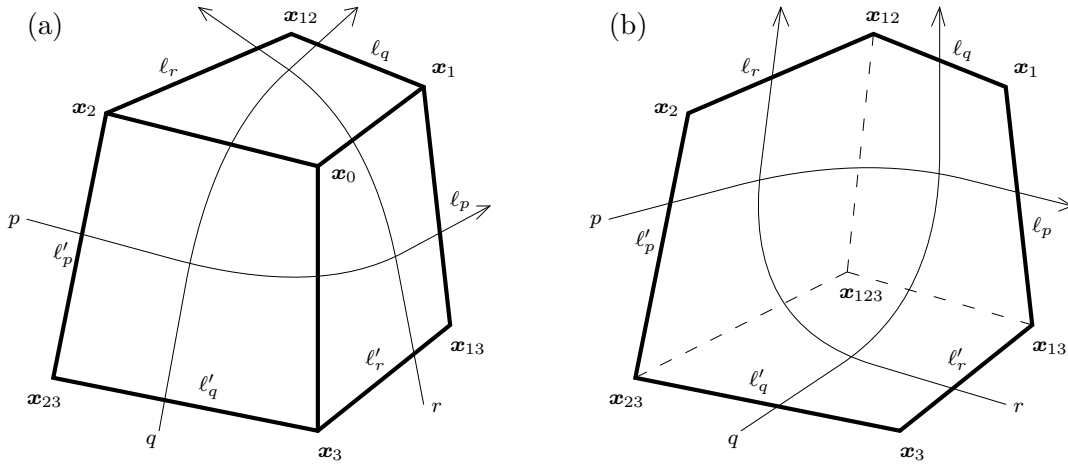


Figure 4: The “front” (a) and “back” (b) faces of the cube in Fig. 1 and “rapidity” lines.

from the top face and moving against the directions of the arrows. Introduce the following three by three matrices

$$X_{pq}(\mathcal{A}) = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{pr}(\mathcal{A}) = \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ C & 0 & D \end{pmatrix}, \quad X_{qr}(\mathcal{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}, \quad (6)$$

where  $A, B, C, D$  are defined in (4) and their dependence on the angles  $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$  is implicitly understood. It follows that

$$(\ell'_p, \ell'_q, \ell'_r)^t = X_{pq}(\mathcal{A}_1) X_{pr}(\mathcal{A}_2) X_{qr}(\mathcal{A}_3) (\ell_p, \ell_q, \ell_r)^t \quad (7)$$

where

$$\mathcal{A}_j = \{\alpha_j, \beta_j, \gamma_j, \delta_j\}, \quad j = 1, 2, 3, \quad (8)$$

the lengths  $\ell_p, \ell_q, \dots$  are defined as in Fig.4, and the superscript “ $t$ ” denotes the matrix transposition. Performing similar calculations for the back faces in Fig.4b and equating the resulting three by three matrices, one obtains

$$X_{pq}(\mathcal{A}_1) X_{pr}(\mathcal{A}_2) X_{qr}(\mathcal{A}_3) = X_{qr}(\mathcal{A}'_3) X_{pr}(\mathcal{A}'_2) X_{pq}(\mathcal{A}'_1). \quad (9)$$

where

$$\mathcal{A}'_j = \{\alpha'_j, \beta'_j, \gamma'_j, \delta'_j\}, \quad j = 1, 2, 3. \quad (10)$$

This matrix relation contains exactly nine scalar equation where LHS only depends on the front angles (8), while the RHS only depends on the back angles (10). Solving these equations one can obtain explicit form of the map (3). The resulting expression are rather complicated and not particularly useful. However the mere fact that the the map (3) satisfy a very special Eq.(9) is extremely important. Indeed, rewrite this equation as

$$X_{pq}(\mathcal{A}_1) X_{pr}(\mathcal{A}_2) X_{qr}(\mathcal{A}_3) = \mathcal{R}_{123} \left( X_{qr}(\mathcal{A}_3) X_{pr}(\mathcal{A}_2) X_{pq}(\mathcal{A}_1) \right) \quad (11)$$

where  $\mathcal{R}_{123}$  is an operator acting as the substitution (3) for any function  $F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$  of the angles,

$$\mathcal{R}_{123} \left( F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \right) = F(\mathcal{A}'_1, \mathcal{A}'_2, \mathcal{A}'_3) \quad (12)$$

Then, following the arguments of [19], one can show that the map (3) satisfies the *functional tetrahedron equation* [20]

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} = \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123} , \quad (13)$$

where both sides are compositions of the maps (3), involving six different sets of angles. Algebraically, this equation arises as an associativity condition for the cubic algebra (11). To discuss its geometric meaning we need to introduce *discrete evolution systems* associated with the map (3).

## 2.2 Discrete evolution systems: “Existence as integrability”

Consider a sub-lattice  $L$  of the 3D quadrilateral lattice, which only includes points  $\mathbf{x}(m)$  with  $m_1, m_2, m_3 \geq 0$ . The boundary of this sub-lattice is a 2D discrete surface formed by quadrilaterals with the vertices  $\mathbf{x}(m)$  having at least one of their integer coordinates  $m_1, m_2, m_3$  equal to zero and the other two non-negative. Assume that all quadrilateral angles on this surface are known, and consider them as initial data. Then repeatedly applying the map (3) one can calculate angles on all faces of the sub-lattice  $L$ , defined above (one has to start from the corner  $\mathbf{x}(0)$ ). The process can be visualized as an evolution of the initial data surface where every transformation (3) corresponds to a “flip” between the front and back faces (Fig. 2) of some cube adjacent to the surface. This makes the surface looking as a 3D “staircase” (or a pile of cubes) in the intersection corner of the three coordinate planes, see Fig. 5 showing two stages of this process. Note, that the corresponding evolution equations can be written in a covariant form for an arbitrary lattice cube (see Eq.(26) below for an example). It is also useful to have in mind that the above evolution can be defined purely geometrically as a *ruler-and-compass* type construction. Indeed the construction of the point  $\mathbf{x}_{123}$  in Fig. 1 from the points  $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{23}$  (and that is what is necessary for flipping a cube) only requires a *2D-ruler* which allows to draw planes through any three non-collinear points in the Euclidean space.

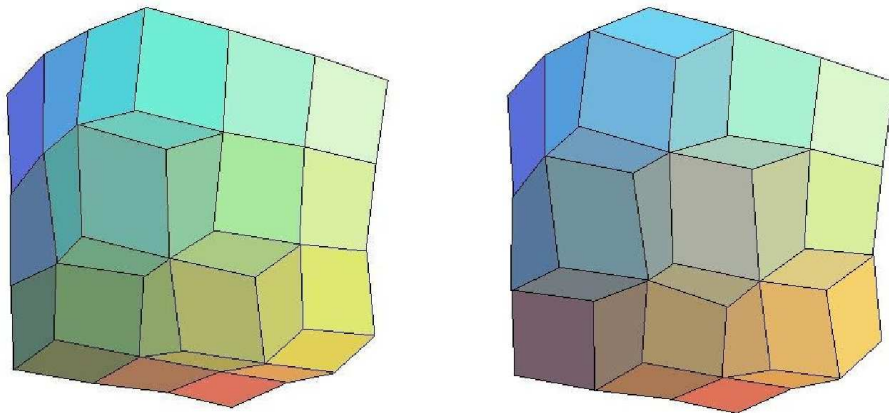


Figure 5: Visualization of the 3D “staircase” evolution.

Similar evolution systems can be defined for other quadrilateral lattices instead of the 3D cubic lattice considered above. Since the evolution is local (only one cube is flipped at a time) one could consider finite lattices as well. For example, consider six adjacent quadrilateral faces

covering the front surface of the rhombic dodecahedron<sup>2</sup> shown in Fig. 6. Suppose that all angles on these faces are given and consider them as initial data. Now apply a sequence of four maps (3) and calculate angles on the back surface of the rhombic dodecahedron. This can be done in two alternative ways, corresponding to the two different dissection of the rhombic dodecahedron into four cubes shown in Fig.6. The functional tetrahedron equation (13) states

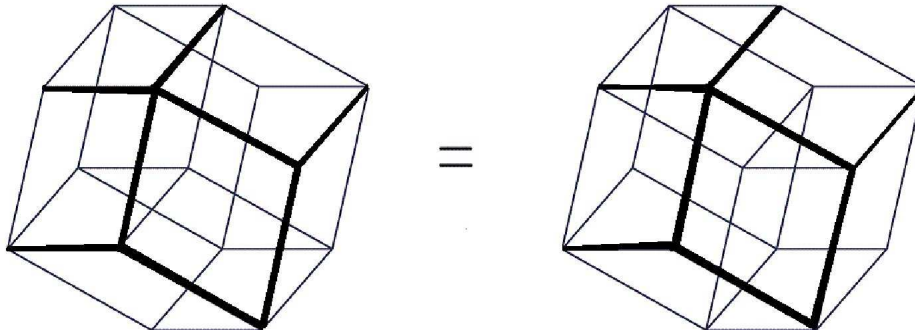


Figure 6: Two dissections of the rhombic dodecahedron into four quadrilateral hexahedra.

that the results will be the same. Thereby it gives an algebraic proof for the equivalence of two “ruler-and-compass” type constructions of the back surface of the dodecahedron in Fig. 6. Can we also prove this equivalence geometrically? Although from the first sight this does not look trivial, it could be easily done from the point of view of the 4D geometry. The required statement follows just from the fact of existence of the quadrilateral lattice with the topology of the 4D cube [25]. The latter is defined by eight intersecting 3-planes in a general position in the 4-space. The two rhombic dodecahedra shown in Fig. 6 are obtained by a dissection of the 3-surface of this 4-cube, along its 2-faces, so these dodecahedra must have exactly the same quadrilateral 2-surface. Thus the functional tetrahedron equation (13), which plays the role of integrability condition for the discrete evolution system associated with the map (3), simply follows from the mere fact of existence of the 4-cube, which is the simplest 4D quadrilateral lattice.

Note that, to our knowledge, the linear problem (4) for the lengths of the quadrilateral with coefficients depending on the angles was not hitherto considered. Much attention was given to the linear problem for coordinates of lattice vertices introduced in [25]. In the notations of Fig. 3 it reads

$$\mathbf{x}_{23} - \mathbf{x}_0 = \frac{\sin \beta}{\sin \alpha} \frac{\ell'_p}{\ell_p} (\mathbf{x}_2 - \mathbf{x}_0) + \frac{\sin \gamma}{\sin \alpha} \frac{\ell'_q}{\ell_q} (\mathbf{x}_3 - \mathbf{x}_0) \quad (14)$$

where  $\mathbf{x}_0, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_{23}$  are 3D coordinate of the vertices. The corresponding evolution equations [25,34] mix lengths and angles in a complicated way. To the contrary, our map (3) only involve angle variables. It describes internal geometry of the quadrilateral lattice independently of its embedding into the target space (the latter, of course, depends on the length variables).

Finally, note that the same map (3) also arises if one considers a linear problem for unit vectors associated with directions of the lattice edges. These vectors satisfy the relation (in the notation of Fig.3)

$$\begin{pmatrix} \mathbf{v}'_p \\ \mathbf{v}'_q \end{pmatrix} = (X^{-1})^t \begin{pmatrix} \mathbf{v}_p \\ \mathbf{v}_q \end{pmatrix} \quad (15)$$

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<sup>2</sup>It is worth noting that the most general rhombic dodecahedron with quadrilateral faces can only be embedded into (at least) the 4D Euclidean space.

where  $X$  is the two by two matrix defined in (4) and

$$\mathbf{v}_p = \frac{\mathbf{x}_3 - \mathbf{x}_0}{\ell_p}, \quad \mathbf{v}_q = \frac{\mathbf{x}_0 - \mathbf{x}_2}{\ell_q}, \quad \mathbf{v}'_p = \frac{\mathbf{x}_{23} - \mathbf{x}_2}{\ell'_p}, \quad \mathbf{v}'_q = \frac{\mathbf{x}_3 - \mathbf{x}_{23}}{\ell'_q}. \quad (16)$$

The property (5) follows from the condition  $\mathbf{v}^2 = 1$ .

### 3 Variational principle for the 3D circular lattices

#### 3.1 Poisson structure of circular lattices

The 3D *circular lattice* [31] is a special 3D quadrilateral lattice where all faces are circular quadrilaterals (i.e., quadrilaterals which can be inscribed into a circle). The existence of these lattices is established by the following beautiful geometry theorem due to Miquel [35] (see Fig. 7)

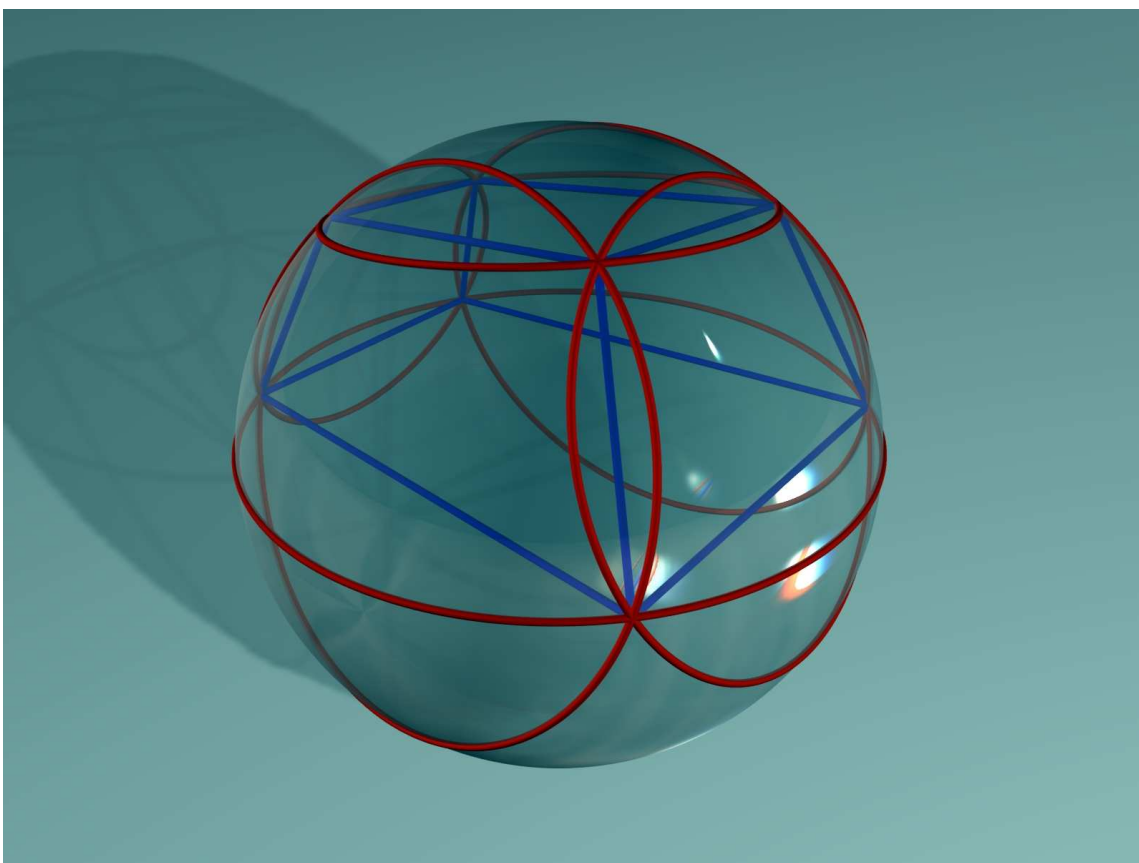


Figure 7: Miquel configuration of circles in 3D space, an elementary hexahedron and its circumsphere.

**Miquel theorem.** Consider four points  $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  in general position in  $\mathbb{R}^N$ ,  $N \geq 3$ . On each of the three circles  $c(\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j)$ ,  $0 \leq i < j \leq 3$  choose an additional new point  $\mathbf{x}_{ij}$ . Then there exist a unique point  $\mathbf{x}_{123}$  which simultaneously belongs to the three circles  $c(\mathbf{x}_1, \mathbf{x}_{12}, \mathbf{x}_{13})$ ,  $c(\mathbf{x}_2, \mathbf{x}_{12}, \mathbf{x}_{23})$  and  $c(\mathbf{x}_3, \mathbf{x}_{13}, \mathbf{x}_{23})$ .

It is easy to see that the above six circles lie on the same sphere. It follows then that every elementary “cube” on a circular lattice (whose vertices are at the circle intersection points) is

inscribable into a sphere, see Fig. 7. The general formulae of the previous subsection can be readily specialized for the circular lattices. A circular quadrilateral has only two independent angles. In the notation of Fig. 3 one has

$$\gamma = \pi - \beta, \quad \delta = \pi - \alpha. \quad (17)$$

Due to the Miquel theorem we can simply impose these restrictions on all faces of the lattice without running to any contradictions. The two by two matrix in (4) takes the form

$$X = \begin{pmatrix} k & a^* \\ -a & k \end{pmatrix}, \quad \det X = 1, \quad (18)$$

where we have introduced new variables

$$k = (\csc \alpha) \sin \beta, \quad a = (\csc \alpha) \sin(\alpha + \beta), \quad a^* = (\csc \alpha) \sin(\alpha - \beta), \quad (19)$$

instead of the two angles  $\{\alpha, \beta\}$ . Note that the new variables are constrained by the relation

$$aa^* = 1 - k^2. \quad (20)$$

Conversely, one has

$$\cos \alpha = \frac{a - a^*}{2k}, \quad \cos \beta = \frac{a + a^*}{2}. \quad (21)$$

Let the variables  $\{k_j, a_j, a_j^*\}$ ,  $\{k'_j, a'_j, a_j^{*'}\}$ ,  $j = 1, 2, 3$ , correspond to the front and back faces of the cube. The map (3) then read explicitly

$$\mathcal{R}_{123} : \begin{cases} (k_2 a_1^*)' = k_3 a_1^* - \varepsilon k_1 a_2^* a_3, & (k_2 a_1)' = k_3 a_1 - \varepsilon k_1 a_2 a_3^*, \\ (a_2^*)' = a_1^* a_3^* + \varepsilon k_1 k_3 a_2^*, & (a_2)' = a_1 a_3 + \varepsilon k_1 k_3 a_2, \\ (k_2 a_3^*)' = k_1 a_3^* - \varepsilon k_3 a_1 a_2^*, & (k_2 a_3)' = k_1 a_3 - \varepsilon k_3 a_1^* a_2, \end{cases} \quad (22)$$

where  $\varepsilon = +1$  and

$$k'_2 = \sqrt{1 - a'_2 a_2^{*'}}. \quad (23)$$

At this point we note that exactly the same map together with the corresponding equations (9) and (13) were previously obtained in [21]. Moreover, it was discovered that this map is a canonical transformation preserving the Poisson algebra

$$\{a_i, a_j^*\} = 2 \delta_{ij} k_i^2, \quad \{k_i, a_j\} = \delta_{ij} k_i a_i, \quad \{k_i, a_j^*\} = -\delta_{ij} k_i a_i^*, \quad i, j = 1, 2, 3, \quad (24)$$

where  $k_i^2 = 1 - a_i a_i^*$ . Note that variables  $k, a, a^*$  on different quadrilaterals are in involution. The same Poisson algebra in terms of angle variables reads

$$\{\alpha_i, \beta_j\} = \delta_{ij}, \quad \{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0. \quad (25)$$

This ‘‘ultra-local’’ symplectic structure trivially extends to any circular quad-surface of initial data, discussed above. To resolve an apparent ambiguity in naming of the angles, this surface must be equipped with the rapidity lines, similar to those in Fig. 4<sup>3</sup>. In addition, the angles for

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<sup>3</sup>We refer the reader to our previous paper [6] where the relationship between the rapidity graphs and quadrilateral lattices is thoroughly discussed.

each quadrilateral should be arranged as in Fig. 3. Then one can assume that the indices  $i, j$  in (25) refer to all quadrilateral on this surface.

Thus, the evolution defined by the map (22) is a symplectic transformation. The corresponding equations of motion for the whole lattice (the analog of the Hamilton-Jacobi equations) can be written in a ‘‘covariant’’ form. For every cube define

$$A_{32} = a_1, \quad A_{23} = a_1^*, \quad A_{31} = a_2', \quad A_{13} = a_2'^*, \quad A_{21} = a_3, \quad A_{12} = a_3^* \quad (26)$$

where  $A_{jk}$  stands for  $A_{jk}(m)$ , where  $m$  is such that  $\mathbf{x}(m)$  coincides with the coordinates of the top front corner of the cube (vertex  $\mathbf{x}_0$  in Fig.1). Let  $T_k$  be the shift operator  $T_k A_{ij}(m) = A_{ij}(m + e_k)$ . Then

$$\tilde{T}_k A_{ij} = \frac{A_{ij} - A_{ik} A_{kj}}{K_{ik} K_{kj}}, \quad K_{ij} = K_{ji} = \sqrt{1 - A_{ij} A_{ji}}, \quad (27)$$

where  $(i, j, k)$  is an arbitrary permutation of  $(1, 2, 3)$  and

$$\tilde{T}_1 = T_1, \quad \tilde{T}_2 = T_2^{-1}, \quad \tilde{T}_3 = T_3. \quad (28)$$

Note that Eq.(27) also imply

$$(\tilde{T}_k K_{ij}) K_{kj} = (\tilde{T}_i K_{kj}) K_{ij}. \quad (29)$$

### 3.2 Variational principle: the Lagrangian and action.

Every canonical transformation

$$P_i, Q_i \rightarrow P_i', Q_i' : \quad dP_i \wedge dQ_i = dP_i' \wedge dQ_i' \quad (30)$$

defines a generating function  $\mathcal{L}(Q, Q')$ :

$$P_i = \frac{\partial \mathcal{L}}{\partial Q_i}, \quad P_i' = -\frac{\partial \mathcal{L}}{\partial Q_i'} : \quad d\mathcal{L} = P_i dQ_i - P_i' dQ_i'. \quad (31)$$

For discrete (in time) canonical transformations, this generating function coincides with the Lagrangian density of the action functional. There are many ways to choose canonical variables; the resulting Lagrangians differ by equivalence transformations. A convenient choice is the variables  $\log k_i$  and  $\frac{1}{2} \log v_i$ ,

$$v_i = \frac{a_i^*}{a_i}, \quad \{ \log k_i, \log v_j \} = 2 \delta_{ij}. \quad (32)$$

The Lagrangian density  $\mathcal{L} = \mathcal{L}(v, v')$  is defined as

$$d\mathcal{L}(v, v') = \frac{1}{2} \sum_{j=1}^3 (\log k_j d \log v_j - \log k_j' d \log v_j'), \quad (33)$$

where  $v = (v_1, v_2, v_3)$  and similarly for  $v'$ . The equations of motion (22) imply the relations

$$k_1'^2 = \frac{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}, \quad k_2^2 = \frac{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}, \quad k_3^2 = \frac{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}, \quad (34)$$

$$k_1'^2 = \frac{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}, \quad k_2'^2 = \frac{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}, \quad k_3'^2 = \frac{\left(1 - \frac{v_2}{v_1 v_3}\right) \left(1 - \frac{v_2'}{v_1 v_3}\right)}{\left(1 - \frac{v_2'}{v_1 v_3}\right) \left(1 - \frac{v_2}{v_1 v_3}\right)}.$$

Introduce new variables

$$\begin{aligned}
e^{-2\Omega_2} &= \frac{v'_2}{v'_1 v'_3}, & e^{-2\Omega_1} &= \frac{v_2}{v_1 v'_3}, & e^{-2\Omega_0} &= \frac{v'_2}{v_1 v_3}, & e^{-2\Omega_3} &= \frac{v_2}{v'_1 v_3}, \\
e^{2\Omega'_2} &= \frac{v_2}{v_1 v_3}, & e^{2\Omega'_1} &= \frac{v'_2}{v'_1 v_3}, & e^{2\Omega'_0} &= \frac{v_2}{v'_1 v'_3}, & e^{2\Omega'_3} &= \frac{v'_2}{v_1 v'_3},
\end{aligned} \tag{35}$$

such that

$$\Omega'_k = \Omega_k - (\Omega_0 + \Omega_1 + \Omega_2 + \Omega_3)/2, \quad k = 0, 1, 2, 3. \tag{36}$$

Remarkably, the differential (33) depends on four independent variables only. Indeed rewriting  $d\mathcal{L}(v, v')$  in terms of six independent variables  $v_1, v_3, \Omega_0, \Omega_1, \Omega_2, \Omega_3$ , one can easily see that all its dependence on  $v_1, v_3$  drops out,

$$d\mathcal{L}(\Omega) = \frac{1}{4} \sum_{j=0}^3 \log \left( \frac{\sinh^2 \Omega_j \sinh^2 \Omega'_j}{\sinh \Omega'_0 \sinh \Omega'_1 \sinh \Omega'_2 \sinh \Omega'_3} \right) d\Omega_j. \tag{37}$$

Integrating the last equation, one obtains

$$\mathcal{L}(v, v') = \frac{1}{2} \sum_{k=0}^3 \left( \Lambda(\Omega_k) + \Lambda(\Omega'_k) \right), \tag{38}$$

where  $\Lambda(x)$  is the modified Lobachevsky function,

$$\Lambda(x) = \int_0^x \log |2 \sinh y| dy. \tag{39}$$

The variables  $\Omega_k, \Omega'_k$  in the RHS of (38) are understood as functions of  $v, v'$ , defined in (35).

In the above derivations of the equations of motion (34) and the Lagrangian density (38), we considered an isolated cube and the associated face variables  $\{k_j, v_j\}, \{k'_j, v'_j\}, j = 1, 2, 3$ . Restoring now the coordinates  $m \in \mathbb{Z}^3$  for the whole lattice (see the definitions before Fig.1),

$$\begin{aligned}
v_j &\rightarrow v_j(m), & v'_j &\rightarrow v'_j(m + e_j), \\
k_j &\rightarrow k_j(m), & k'_j &\rightarrow k'_j(m + e_j)
\end{aligned} \quad j = 1, 2, 3 \tag{40}$$

one has for the total action

$$\mathcal{S}^{(cl)}(\{v\}) = \sum_{m \in \mathbb{Z}^3} \mathcal{L}(v(m), v'(m)). \tag{41}$$

where  $v'_j(m) = v_j(m + e_j)$ . The variation principle for this action leads to the Lagrangian equations of motion with respect to the variables  $v_j(m)$ . To write them in a covariant form it is convenient to define new variables

$$u_1(m) = v_1(m), \quad u_2(m) = 1/v_2(m), \quad u_3(m) = v_3(m). \tag{42}$$

For an arbitrary lattice site  $m \in \mathbb{Z}^3$  denote

$$u_i^{(0)} \equiv u_i(m), \quad u_i^{(j)} \equiv u_i(m + e_j), \quad u_i^{(jk)} \equiv u_i(m + e_j + e_k), \tag{43}$$

where the indices  $i, j, k$  independently take any of the three values 1, 2, 3. Then the Lagrangian equations of motion, determining the stationary point of the action (41) take the form

$$\frac{(1 - u_i^{(i)} u_j^{(ij)} u_k^{(i)}) (1 - u_i^{(i)} u_j^{(i)} u_k^{(ik)})}{(1 - u_i^{(i)} u_j^{(i)} u_k^{(i)}) (1 - u_i^{(i)} u_j^{(ij)} u_k^{(ik)})} = \frac{(1 - u_i^{(i)} u_j^{(j)} u_k^{(0)}) (1 - u_i^{(i)} u_j^{(0)} u_k^{(k)})}{(1 - u_i^{(i)} u_j^{(0)} u_k^{(0)}) (1 - u_i^{(i)} u_j^{(j)} u_k^{(k)})}, \quad (44)$$

where  $(i, j, k)$  is any permutation of  $(1, 2, 3)$ . Note that similar, but different, equations of motion arose previously in [32, 36].

It is sometimes convenient to use the  $k$ -variables,  $k_j(m)$ , instead of the  $v$ -variables. The corresponding Lagrangian is obtained from (38) with a Legendre transform

$$\bar{\mathcal{L}}(k, k') = -\frac{1}{2} \sum_{j=1}^3 \log k_j \log v_j + \mathcal{L}(v, v') + \frac{1}{2} \sum_{j=1}^3 \log k'_j \log v'_j. \quad (45)$$

where the variables  $v, v'$  are now considered as functions of  $k, k'$  implicitly defined by (34). The total action

$$\mathcal{S}^{(cl)}(\{k\}) = \sum_{m \in \mathbb{Z}^3} \bar{\mathcal{L}}(k(m), k'(m)). \quad (46)$$

obviously coincides with (41), up to boundary terms. Note that the Lagrangian  $\mathcal{L}(v, v')$  has a gauge symmetry: it depends only on four combinations of the six independent variables  $v_j, v'_j$  (see (35,36) above). As a result the variables  $k_j, k'_j$  in  $\bar{\mathcal{L}}(k, k')$  are constrained by two relations

$$k_1 k_2 = k'_1 k'_2, \quad k_2 k_3 = k'_2 k'_3, \quad (47)$$

which correspond to (29). Here we prefer to avoid Lagrangian multipliers and deal with the restricted configuration space directly. The local constraint (47), taken for all lattice sites, leads to certain restrictions for boundary values of  $k$ 's and makes the Lagrangian equations of motion more complicated than those in the  $v$ -variables. For instance, the local equations of motion in the  $k$ -variables involves eight adjacent hexahedra, while those in  $v$ -variables, given by (44), involve only two. Nevertheless, the Dirichlet problem for the action (46) is well posed [33].

Note also that the logarithmic terms in (45) can be written as

$$-\sum_{j=1}^3 \log k_j \log v_j + \sum_{j=1}^3 \log k'_j \log v'_j = -\sum_{j=0}^3 \left( \Omega_j \log(2 \sinh \Omega_j) + \Omega'_j \log(2 \sinh \Omega'_j) \right). \quad (48)$$

## 4 Quantization.

### 4.1 Tetrahedron equation

In this section we construct two different, but related quantizations of the map (22) and obtain two solutions of the full quantum tetrahedron equation (see Eq.(57) below). In both cases we start with the canonical quantization of the Poisson algebra (25),

$$[\alpha_i, \beta_j] = \xi \hbar \delta_{ij}, \quad [\alpha_i, \alpha_j] = 0, \quad [\beta_i, \beta_j] = 0, \quad (49)$$

where  $\hbar$  is the quantum parameter (the Planck constant) and  $\xi$  is a numerical coefficient, introduced for a further convenience. The indices  $i, j$  label the faces of the ‘‘surface of initial data’’ discussed above. Since the commutation relations (49) are ultra-local (in the sense that

the angle variables on different faces commute with each other), let us concentrate on the local Heisenberg algebra,

$$\mathbf{H} : \quad [\alpha, \beta] = \xi \hbar , \quad (50)$$

for a single lattice face (remind that the angles shown in Fig. 3 are related by (17)). The map (22) contains the quantities  $k, a, a^*$ , defined in (19), which now become operators. For definiteness, assume that the non-commuting factors in (19) are ordered exactly as written. Then the definitions (19) give

$$\begin{aligned} k &= (U - U^{-1})^{-1} (V - V^{-1}), \\ a &= q^{-\frac{1}{2}} (U - U^{-1})^{-1} (U V - U^{-1} V^{-1}), \\ a^* &= q^{+\frac{1}{2}} (U - U^{-1})^{-1} (U V^{-1} - U^{-1} V), \end{aligned} \quad (51)$$

where the elements  $U$  and  $V$  generate the Weyl algebra,

$$U V = q V U, \quad U = e^{i\alpha}, \quad V = e^{i\beta}, \quad (52)$$

with

$$q = e^{-\xi \hbar}. \quad (53)$$

The operators (51) obey the commutation relations of the  $q$ -oscillator algebra,

$$\text{Osc}_q : \quad \begin{cases} q a^* a - q^{-1} a a^* = q - q^{-1}, & k a^* = q a^* k, & k a = q^{-1} a k, \\ k^2 = q (1 - a^* a) = q^{-1} (1 - a a^*). \end{cases} \quad (54)$$

This algebra is, obviously, a quantum counterpart of the Poisson algebra (24). In the previous Section we have already mentioned the result of [21] that

- (i) the map (22) is an automorphism of the tensor cube of the Poisson algebra (24) (remind that the relation (20) should be taken into account in (22)).

In the same paper [21] it was also shown that

- (ii) there exists a quantum version of the map (22), which acts as an automorphism of the tensor cube of the  $q$ -oscillator algebra (54). The formulae (22) for the quantum map stay exactly the same, but the relation (20) should be replaced by either of the two relations on the second line of (54), for instance,  $k^2 = q (1 - a^* a)$ . In particular, (23) should be replaced with

$$(k'_2)^2 = q (1 - a_2^{*'} a_2') . \quad (55)$$

For irreducible representations of the  $q$ -oscillator algebra (54) the formulae (22) and (55) uniquely determine the  $\mathcal{R}_{123}$  as an internal automorphism,

$$\mathcal{R}_{123}(F) = R_{123} F R_{123}^{-1}, \quad F \in \text{Osc}_q \otimes \text{Osc}_q \otimes \text{Osc}_q . \quad (56)$$

It follows then from (13) that the linear operator  $R$  satisfies the quantum tetrahedron equation

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123} , \quad (57)$$

where the operators  $R_{123}$ ,  $R_{145}$ ,  $R_{246}$  and  $R_{356}$  act as (56) in the three factors of the tensor product of the six algebras  $q$ -oscillator algebras (indicated by their subscripts) and act as the unit operator in the remaining three factors.

## 4.2 Fock space representation model

In this subsection we set  $\xi = 1$  in (49). Assuming  $\hbar > 0$  we have  $q = e^{-\hbar} < 1$ . Define the Fock space representation of a single  $q$ -oscillator algebra (54),

$$F_q : \quad a |0\rangle = 0, \quad |n\rangle = \frac{(a^*)^n}{\sqrt{(q^2, q^2)_n}} |0\rangle, \quad k |n\rangle = q^{n+1/2} |n\rangle, \quad n \geq 0 \quad (58)$$

Then using (22), (54), (55) and (56) one can show that the the matrix elements of  $R$  are given by

$$\begin{aligned} \langle n_1, n_2, n_3 | R | n'_1, n'_2, n'_3 \rangle &= \delta_{n_1+n_2, n'_1+n'_2} \delta_{n_2+n_3, n'_2+n'_3} \sqrt{\frac{(q^2, q^2)_{n'_1} (q^2, q^2)_{n'_2} (q^2, q^2)_{n'_3}}{(q^2, q^2)_{n_1} (q^2, q^2)_{n_2} (q^2, q^2)_{n_3}}} \\ &\times \frac{(-1)^{n_2} q^{(n'_1-n_2)(n'_3-n_2)}}{(q^2; q^2)_{n'_2}} \frac{(q^{2(1-n'_2+n_3)}; q^2)_\infty}{(q^{2(1+n_3)}; q^2)_\infty} {}_2\phi_1(q^{-2n'_2}, q^{2(1+n'_3)}, q^{2(1-n'_2+n_3)}; q^2, q^{2(1+n_1)}), \end{aligned} \quad (59)$$

where

$$(x; q^2)_n = (1-x)(1-q^2x) \cdots (1-q^{2(n-1)}x), \quad (60)$$

and

$${}_2\phi_1(a, b, c; q^2, z) = \sum_{n=0}^{\infty} \frac{(a; q^2)_n (b; q^2)_n}{(q^2; q^2)_n (c; q^2)_n} z^n \quad (61)$$

is the  $q$ -deformed Gauss hypergeometric series. Eq.(56) together with (22) and (55) lead to recurrence relations for the matrix elements of  $R$ . Such relations along with expressions for matrix elements of  $R$  for small values of the occupation numbers  $n_i$ , were obtained in [21] (see Eqs.(30-32) therein). We have now solved those recurrence relations and obtained the explicit formula (59), given above<sup>4</sup>. This 3D  $R$ -matrix satisfies the constant tetrahedron equation (57),

$$\sum_{n'_j=0}^{\infty} R_{n_1, n_2, n_3}^{n'_1, n'_2, n'_3} R_{n'_1, n'_4, n'_5}^{n''_1, n''_4, n''_5} R_{n'_2, n'_4, n'_6}^{n''_2, n''_4, n''_6} R_{n'_3, n'_5, n'_6}^{n''_3, n''_5, n''_6} = \sum_{n'_j=0}^{\infty} R_{n_3, n_5, n_6}^{n'_3, n'_5, n'_6} R_{n_2, n_4, n'_6}^{n'_2, n'_4, n'_6} R_{n_1, n_4, n'_5}^{n'_1, n'_4, n'_5} R_{n'_1, n'_2, n'_3}^{n''_1, n''_2, n''_3} \quad (62)$$

where the sum is taken over six the indices  $n'_1, n'_2, n'_3, n'_4, n'_5, n'_6$  and

$$R_{n_1, n_2, n_3}^{n'_1, n'_2, n'_3} = \langle n_1, n_2, n_3 | R | n'_1, n'_2, n'_3 \rangle. \quad (63)$$

Note that Eq.(62) does not contain any spectral parameters.

Now define a model of lattice field theory. On each edge on the cubic lattice place a fluctuating spin variable  $n$ , taking an infinite number of integer values  $n = 0, 1, 2, \dots \infty$ . To each vertex of the lattice assign a local weight factor  $\langle n_1, n_2, n_3 | R | n'_1, n'_2, n'_3 \rangle$ , given by (59), depending on six spin variables placed on the edges surrounding the vertex, arranged as in Fig.8. The partition function

$$Z_F = \sum_{(\text{spins})} \prod_{(\text{vertices})} \langle n_1, n_2, n_3 | R | n'_1, n'_2, n'_3 \rangle, \quad (64)$$

---

<sup>4</sup>Eq.(59) follows from Eqs.(29,30) of [21] where  $P_{\beta'}(q^{2\alpha}, q^{2\beta}, q^{2\gamma})$  is substituted with

$$P_{\beta'}(q^{2\alpha}, q^{2\beta}, q^{2\gamma}) = \frac{(q^{2(1-\beta'+\gamma)}; q^2)_\infty}{(q^{2(1+\gamma)}, q^2)_\infty} {}_2\phi_1(q^{-2\beta'}, q^{2(1-\beta'+\beta+\gamma)}, q^{2(1-\beta'+\gamma)}; q^2, q^{2(1+\alpha)}).$$

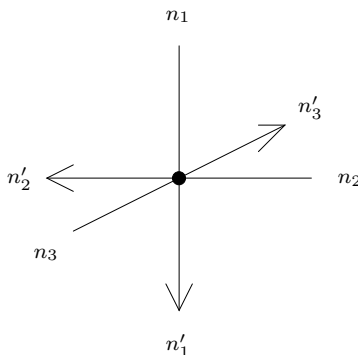


Figure 8: Graphical visualization of three-dimensional  $R$ -matrix (59) and its matrix elements. The orientation of the axes is consistent with Fig. 2

is defined as the sum over all spin configurations of the whole lattice where each configuration is counted with the weight equal to the product of the vertex weights over all lattice vertices. For definiteness we assume fixed boundary condition. The subscript “F” stands for the “Fock representation model”.

In the quasi-classical limit

$$q = e^{-\hbar} \rightarrow 1, \quad \hbar \rightarrow 0, \quad n_i \hbar = -\log k_i = \text{finite}, \quad (65)$$

with large occupation numbers  $n_i, n'_i$ , such that  $k_i$  and  $k'_i$  are kept finite, one obtains

$$\lim_{\hbar \rightarrow 0} \langle n_1, n_2, n_3 | R | n'_1, n'_2, n'_3 \rangle = \exp \left( -\frac{\overline{\mathcal{L}}_F(k, k')}{\hbar} \right), \quad (66)$$

where the arguments of  $\overline{\mathcal{L}}_F(k, k')$  are related by  $k_1 k_2 = k'_1 k'_2$  and  $k_2 k_3 = k'_2 k'_3$ . The function  $\overline{\mathcal{L}}_F(k, k')$  coincides with the classical Lagrangian (45) provided the variables  $k, k'$  in (65) are identified with those in (34). Calculating the integral (64) by the saddle point method, one obtains

$$\log Z_F = -\frac{1}{\hbar} \mathcal{S}^{(cl)} + O(\hbar^0), \quad \hbar \rightarrow 0, \quad (67)$$

where  $\mathcal{S}^{(cl)}$  is the classical action (46) evaluated at its stationary point.

### 4.3 Modular double model

In this subsection we set in (49)

$$\xi = -i, \quad \hbar = \pi b^2, \quad (68)$$

where  $b$  is a free parameter,  $\text{Re } b \neq 0$ . Here it will be more convenient to work with a slightly modified version<sup>5</sup> of the map (22), with the value  $\varepsilon = -1$ . It is worth noting that this modification does not affect the bulk properties of the classical system (41) and leads only to boundary effects. In particular the equations of motion (44) remain unchanged.

Consider a non-compact representation [37] of the  $q$ -oscillator algebra (54) in the space of functions  $f(\sigma) \in \mathbb{L}^2(\mathbb{R})$  on the real line admitting an analytical continuation into an appropriate

<sup>5</sup>Note that the map (22) with  $\varepsilon = \pm 1$  is a particular case a three-parameter map considered in [21].

horizontal strip, containing the real axis in the complex  $\sigma$ -plane (see [37] and [33] for further details). Such representation essentially reduces to that of the Weyl algebra

$$W_q : \quad k w = q w k, \quad q = e^{i\pi b^2}, \quad (69)$$

realized as multiplication and shift operators

$$k |\sigma\rangle = i e^{\pi\sigma b} |\sigma\rangle, \quad w |\sigma\rangle = |\sigma - ib\rangle. \quad (70)$$

The generators  $a, a^*$  in (54) are expressed as

$$a = (1 - q k^2)^{1/2} w^{-1}, \quad a^* = (1 - q^{-1} k^2)^{1/2} w. \quad (71)$$

As explained in [38] the representation (70) is not, in general, irreducible. The reason is that the algebra (69) contains a large center, generated by the elements  $\tilde{k}^2$  and  $\tilde{w}^2$ , where  $\tilde{k}, \tilde{w}$  is the modular dual pair,

$$W_{\tilde{q}} : \quad \tilde{k} \tilde{w} = \tilde{q} \tilde{w} \tilde{k}, \quad \tilde{q} = e^{-i\pi b^{-2}}, \quad (72)$$

acting in the same representation space

$$\tilde{k} |\sigma\rangle = -i e^{\pi\sigma b^{-1}} |\sigma\rangle, \quad \tilde{w} |\sigma\rangle = |\sigma + ib^{-1}\rangle. \quad (73)$$

Therefore, the relation (56) alone does not unambiguously define the linear operator  $R_{123}$  in this case. However, we found that if this relation is complemented by its modular dual

$$\tilde{\mathcal{R}}_{123}(\tilde{F}) = R_{123} \tilde{F} R_{123}^{-1}, \quad \tilde{F} \in \text{Osc}_{\tilde{q}} \otimes \text{Osc}_{\tilde{q}} \otimes \text{Osc}_{\tilde{q}}, \quad (74)$$

then the pair of relations (56) and (74) determine the operator  $R_{123}$  uniquely<sup>6</sup>. The dual  $q$ -oscillator algebra  $\text{Osc}_{\tilde{q}}$  is realized through the dual Weyl pair (72) and the relations

$$\tilde{a} = (1 - \tilde{q} \tilde{k}^2)^{1/2} \tilde{w}^{-1}, \quad \tilde{a}^* = (1 - \tilde{q}^{-1} \tilde{k}^2)^{1/2} \tilde{w}. \quad (75)$$

The dual version of the map  $\tilde{\mathcal{R}}_{123}$  is defined by the same formulae (22), where quantities  $k_j, a_j, a_j^*$ ,  $j = 1, 2, 3$  are replaced by their “tilded” counterparts  $\tilde{k}_j, \tilde{a}_j, \tilde{a}_j^*$ . The value of  $q$  do not, actually, enter the map (22); but needs to be taken into account in the relations between the generators of the  $q$ -oscillator algebra. Thus, the linear operator  $R_{123}$  in this case simultaneously provides the two maps  $\mathcal{R}_{123}$  and  $\tilde{\mathcal{R}}_{123}$  (with  $\varepsilon = -1$ ). The explicit form of this operator is given below.

Denote

$$\eta = \frac{b + b^{-1}}{2}, \quad (76)$$

and define a special function

$${}_2\Psi_2 \left( \begin{matrix} c_1, c_2 \\ c_3, c_4 \end{matrix} \middle| c_0 \right) = \int_{\mathbb{R}} dz e^{2\pi iz(-c_0 - i\eta)} \frac{\varphi(z + \frac{c_1 + i\eta}{2}) \varphi(z + \frac{c_2 + i\eta}{2})}{\varphi(z + \frac{c_3 - i\eta}{2}) \varphi(z + \frac{c_4 - i\eta}{2})}, \quad (77)$$

where  $\varphi$  is the non-compact quantum dilogarithm [10]

$$\varphi(z) = \exp \left( \frac{1}{4} \int_{\mathbb{R} + i0} \frac{e^{-2izx}}{\sinh(xb) \sinh(x/b)} \frac{dx}{x} \right). \quad (78)$$

---

<sup>6</sup>It is worth mentioning similar phenomena in the construction of the  $R$ -matrix [39] for the modular double of the quantum group  $U_q(sl_2)$  and the representation theory of  $U_q(sl_2, \mathbb{R})$  [40].

The values of  $c_1, c_2, c_3, c_4$  are assumed to be such that poles of numerator in the integrand of (77) lie above the real axis, while the zeroes of the denominator lie below the real axis. For other values of  $c_j$  the integral (77) is defined by an analytic continuation. For  $\text{Im } b^2 > 0$  the integral  ${}_2\Psi_2$  can be evaluated by closing the integration contour in the upper half plane. The result reads

$$\begin{aligned} {}_2\Psi_2 \left( \begin{matrix} c_1, c_2 \\ c_3, c_4 \end{matrix} \middle| c_0 \right) &= e^{i\pi(c_0+i\eta)(c_1-i\eta)-i\pi(4\eta^2+1)/12} \frac{\varphi\left(\frac{c_2-c_1}{2} + i\eta\right)}{\varphi\left(\frac{c_3-c_1}{2}\right)\varphi\left(\frac{c_4-c_1}{2}\right)} \\ &\times {}_2\phi_1(-\tilde{q} e^{\pi(c_1-c_3)/b}, -\tilde{q} e^{\pi(c_1-c_4)/b}, \tilde{q}^2 e^{\pi(c_1-c_2)/b}; \tilde{q}^2, -\tilde{q} e^{\pi(c_3+c_4-c_1-c_2+2c_0)/b}) \\ &\times {}_2\phi_1(-q e^{\pi(c_3-c_1)b}, -q e^{\pi(c_4-c_1)b}, q^2 e^{\pi(c_2-c_1)b}; q^2, -q e^{2\pi bc_0}) + (c_1 \leftrightarrow c_2), \end{aligned} \quad (79)$$

where  ${}_2\phi_1$  is defined by (61). This formula is very convenient for numerical calculations.

The kernel of  $R$ -matrix serving the pair of the maps (56) and (74) is given by

$$\begin{aligned} \langle \sigma | R | \sigma' \rangle &= \delta_{\sigma_1+\sigma_2, \sigma'_1+\sigma'_2} \delta_{\sigma_2+\sigma_3, \sigma'_2+\sigma'_3} e^{i\pi(\sigma'_1\sigma'_3+i\eta(\sigma'_1+\sigma'_3-\sigma_2))} \\ &\times \sqrt{\frac{\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)}{\varphi(\sigma'_1)\varphi(\sigma'_2)\varphi(\sigma'_3)}} {}_2\Psi_2 \left( \begin{matrix} \sigma_1 - \sigma_3, -\sigma_1 + \sigma_3 \\ \sigma_1 + \sigma_3, -\sigma'_1 - \sigma'_3 \end{matrix} \middle| \sigma_2 \right) \end{aligned} \quad (80)$$

It satisfies the constant tetrahedron equation (57),

$$\begin{aligned} \int_{\mathbb{R}} d\sigma'_1 \dots d\sigma'_6 R_{\sigma_1, \sigma_2, \sigma_3}^{\sigma'_1, \sigma'_2, \sigma'_3} R_{\sigma'_1, \sigma'_4, \sigma'_5}^{\sigma''_1, \sigma'_4, \sigma'_5} R_{\sigma'_2, \sigma'_4, \sigma'_6}^{\sigma''_2, \sigma'_4, \sigma'_6} R_{\sigma'_3, \sigma'_5, \sigma'_6}^{\sigma''_3, \sigma'_5, \sigma'_6} = \\ = \int_{\mathbb{R}} d\sigma'_1 \dots d\sigma'_6 R_{\sigma_3, \sigma_5, \sigma_6}^{\sigma'_3, \sigma'_5, \sigma'_6} R_{\sigma_2, \sigma_4, \sigma'_6}^{\sigma'_2, \sigma'_4, \sigma'_6} R_{\sigma_1, \sigma'_4, \sigma'_5}^{\sigma'_1, \sigma'_4, \sigma'_5} R_{\sigma'_1, \sigma'_2, \sigma'_3}^{\sigma''_1, \sigma'_2, \sigma'_3} \end{aligned} \quad (81)$$

where

$$\langle \sigma_1, \sigma_2, \sigma_3 | R | \sigma'_1, \sigma'_2, \sigma'_3 \rangle = R_{\sigma'_1, \sigma'_2, \sigma'_3}^{\sigma_1, \sigma_2, \sigma_3} \quad (82)$$

and the integrals are taken along the real line from  $-\infty$  to  $+\infty$ .

Similarly to (64) define a “modular double model”

$$Z_M = \int \dots \int \prod_{(\text{vertices})} \langle \sigma | R | \sigma' \rangle \prod_{(\text{edges})} d\sigma, \quad (83)$$

where the edge spins  $\sigma$  now take continuous values on the real line. We assume fixed boundary conditions. Note that due to the presence of two delta-functions in (80) the edge spins are constrained by two relations

$$\sigma_1 + \sigma_3 = \sigma'_1 + \sigma'_3, \quad \sigma_2 + \sigma_3 = \sigma'_2 + \sigma'_3, \quad (84)$$

at each vertex of the lattice.

Consider the quasi-classical limit of the  $R$ -matrix (80)

$$b \rightarrow 0, \quad |\sigma_j|, |\sigma'_j| \rightarrow \infty, \quad (85)$$

such that the variables

$$k_j = ie^{\pi b \sigma_j}, \quad k'_j = ie^{\pi b \sigma'_j}, \quad j = 1, 2, 3, \quad (86)$$

are kept finite. Anticipating a connection with the classical formula (45) let us identify the variables (86) with those in (34). Further, for real  $b$  the squares  $k_j^2$  and  $k'_j{}^2$  are negative (notice the factor  $i$  in front of the exponents in (86)). To accommodate this extra minus sign we will use a slightly modified form of (35), namely,

$$\begin{aligned} e^{2i\omega_2} &= \frac{v'_2}{v'_1 v'_3}, & e^{2i\omega_1} &= \frac{v_2}{v_1 v_3}, & e^{2i\omega_0} &= \frac{v'_2}{v_1 v_3}, & e^{2i\omega_3} &= \frac{v_2}{v'_1 v_3}, \\ e^{-2i\omega_2} &= \frac{v_2}{v_1 v_3}, & e^{-2i\omega_1} &= \frac{v'_2}{v'_1 v_3}, & e^{-2i\omega_0} &= \frac{v_2}{v'_1 v'_3}, & e^{-2i\omega_3} &= \frac{v'_2}{v_1 v'_3}. \end{aligned} \quad (87)$$

where the new variables  $\omega_j$  and  $\omega'_j$  obey the relations

$$\omega'_k = \frac{\pi}{2} + \omega_k - (\omega_0 + \omega_1 + \omega_2 + \omega_3)/2, \quad k = 0, 1, 2, 3. \quad (88)$$

Note also a useful identity

$$\omega_i + \omega_j + \omega'_k + \omega'_l = \pi, \quad (i, j, k, l) = \text{perm}(0, 1, 2, 3), \quad (89)$$

where  $(i, j, k, l)$  is any permutation of  $(0, 1, 2, 3)$ . If the  $\omega$ -variables lie in the domain

$$0 < \omega_k < \pi, \quad 0 < \omega'_k < \pi, \quad k = 0, 1, 2, 3, \quad (90)$$

then in the quasi-classical limit (85) the integral (77) entering the formula (80) can be estimated by the saddle point method

$$\langle \sigma | R | \sigma' \rangle = \exp \left\{ -\frac{\bar{\mathcal{L}}_{\mathcal{M}}(k, k')}{\pi b^2} + O(1) \right\}, \quad b \rightarrow 0. \quad (91)$$

In practice, it is easier to calculate logarithmic derivatives of (80) with respect variables  $k_j$  and  $k'_j$ , rather than the expression (80) itself. In this way one obtains

$$d\bar{\mathcal{L}}_{\mathcal{M}}(k, k') = \frac{1}{2} \sum_{j=0}^3 (\omega_j d \log \sin \omega_j + \omega'_j d \log \sin \omega'_j) - \frac{\pi}{2} (d \log \sin \omega_2 + d \log \sin \omega'_2). \quad (92)$$

It follows then

$$\bar{\mathcal{L}}_{\mathcal{M}}(k, k') = \mathcal{L}_{\mathcal{M}}(v, v') + \text{Legendre terms}, \quad (93)$$

where the non-trivial part

$$\mathcal{L}_{\mathcal{M}}(v, v') = \frac{1}{2} \sum_{j=0}^3 (\mathcal{L}(\omega_j) + \mathcal{L}(\omega'_j)), \quad (94)$$

is expressed in terms of the Lobachevsky function [41],

$$\mathcal{L}(\omega) = - \int_0^\omega \log 2 \sin x \, dx, \quad 0 < \omega < \pi, \quad (95)$$

while the Legendre terms read

$$\text{Legendre terms} = \frac{1}{2} \sum_{j=0}^3 (\omega_j \log 2 \sin \omega_j + \omega'_j \log 2 \sin \omega'_j) - \frac{\pi}{2} (\log 2 \sin \omega_2 + \log 2 \sin \omega'_2) . \quad (96)$$

Note, that in terms of the variables  $k, k', v, v'$  (connected with  $\omega$ 's by (34) and (87)) the differential of (94) reads

$$d\mathcal{L}_M(v, v') = \frac{1}{2i} \sum_{j=1}^3 \left( \log \frac{k_j}{i} d \log v_j - \log \frac{k'_j}{i} d \log v'_j \right) , \quad (97)$$

while the Legendre terms take the form

$$\text{Legendre terms} = -\frac{1}{2i} \sum_{j=1}^3 \left( \log \frac{k_j}{i} \log v_j - \log \frac{k'_j}{i} \log v'_j \right) . \quad (98)$$

The formulae (97) and (98) are in a complete agreement with (33) and (45), except the overall normalization and the replacement  $k_j \rightarrow k_j/i$  and  $k'_j \rightarrow k'_j/i$  which is a gauge transformation resulted from a different choice of the canonical variables. The total action

$$\mathcal{S}_M^{(cl)}(\{k\}) = \sum_{m \in \mathbb{Z}^3} \mathcal{L}_M(k(m), k'(m)) \quad (99)$$

coincide (to within an overall factor  $i^{-1}$  and boundary terms) with an analytic continuation of (46) into a regime where the variables  $k_j^2$  are negative and the variables  $v_j$  are unimodular. Substituting (91) into (83) one obtains

$$\log Z_M = -\frac{1}{\pi b^2} \mathcal{S}_M^{(cl)} + O(1), \quad b \rightarrow 0 \quad (100)$$

where  $\mathcal{S}_M^{(cl)}$  is the classical action (99) evaluated at its stationary point. It is obvious from (94) that this expression is real.

Before concluding this section let us make a remark about the spatial symmetry properties of the modular model. It is convenient to introduce another kernel,

$$\langle \sigma | \tilde{R} | \sigma' \rangle = \langle \sigma | R | \sigma' \rangle e^{\pi \eta (\sigma_1 - \sigma_2 + \sigma_3 + \sigma'_1 - \sigma'_2 + \sigma'_3) / 2} \quad (101)$$

differing from (80) by a phase factor which introduces particular “external fields” but does not destroy the integrability of the model. The new kernel  $\tilde{R}$  obeys simple relations

$$\tilde{R}_{\sigma_1, \sigma_2, \sigma_3}^{\sigma'_1, \sigma'_2, \sigma'_3} = \tilde{R}_{\sigma_3, \sigma_2, \sigma_1}^{\sigma'_2, \sigma'_2, \sigma'_1} = \tilde{R}_{-\sigma'_2, \sigma'_1, -\sigma'_3}^{-\sigma_2, \sigma'_1, -\sigma_3} . \quad (102)$$

which generate the whole the cube symmetry group.

#### 4.4 The “interaction-round-a-cube” formulation of the modular model

An inconvenient feature of the modular representation model formulated above is that the edge spins are constrained by the relations (84). Here we re-formulate this model in terms of unconstrained *corner* spins, which also take continuous values on the real line. Fig. 9 shows an elementary cube of the lattice with the corner spins “ $a, b, c, d, e, f, g, h$ ” arranged in the same way as in [15]. The corresponding Boltzmann weight reads

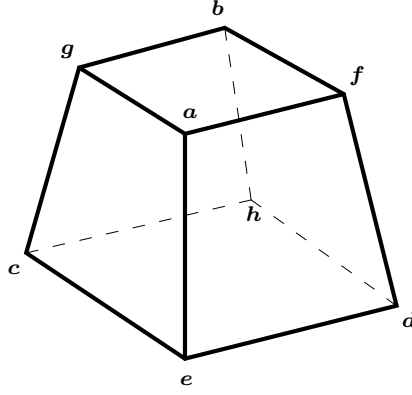


Figure 9: The arrangement of corner spins around a cube.

$$\begin{aligned}
W(a|e, f, g|b, c, d|h; \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= \exp \left[ \sum_{i=1}^3 \mathbf{a}_i (\sigma_i + \sigma'_i) \eta (-1)^{\delta_{i,3}} \right] \times \\
&\times e^{-i\pi(\sigma_1\sigma_2 + i\eta(\sigma'_3 - \sigma_1 - \sigma_2))} \sqrt{\frac{\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)}{\varphi(\sigma'_1)\varphi(\sigma'_2)\varphi(\sigma'_3)}} {}_2\Psi_2 \left( \begin{matrix} \sigma'_1 + \sigma'_2, -\sigma_1 - \sigma_2 \\ \sigma_1 - \sigma_2, -\sigma_1 + \sigma_2 \end{matrix} \middle| -\sigma'_3 \right)
\end{aligned} \quad (103)$$

where  $\sigma_i, \sigma'_i$  are defined as

$$\begin{aligned}
\sigma_1 &= g + f - a - b - \mathbb{T}_1, & \sigma_2 &= e + g - a - c - \mathbb{T}_2, & \sigma_3 &= a + d - e - f + \mathbb{T}_3, \\
\sigma'_1 &= c + d - e - h - \mathbb{T}_1, & \sigma'_2 &= b + d - f - h - \mathbb{T}_2, & \sigma'_3 &= g + h - b - c + \mathbb{T}_3
\end{aligned} \quad (104)$$

and satisfy the constrains (84). The parameters  $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$  are analogs of the spectral parameters in the Zamolodchikov model [14] and its generalization for an arbitrary number  $N \geq 2$  of spin states [16]. They are related by  $\mathbb{T}_j = \log[\tan(\theta_j/2)]$ , where  $\theta_j, j = 1, 2, 3$ , are the dihedral angles in the Zamolodchikov model (they are usually considered as angles of a spherical triangle). The parameters  $\mathbf{a}_i$  are similar to the linear angles in Zamolodchikov model (the side lengths of the above spherical triangle). However, unlike the Zamolodchikov model, the two sets of parameters  $\mathbb{T}_i$  and  $\mathbf{a}_i$  in our case are totally independent. The parameters  $\mathbf{a}_i$  enter only the exponential prefactor in (103); they play the role of the external fields in the model.

Consider spatial symmetry properties of the weight function (103). The group of cube is generated by two elements:  $\tau$  (the reflection with respect to the diagonal plane  $(a, d, h, g)$  in Fig. 9) and  $\rho$  (the  $90^\circ$  rotation around the vertical axis). They act on the spin variables and parameters  $\mathbb{T}_i, \mathbf{a}_i$  as follows

$$\begin{aligned}
\tau &: (a|efg|bcd|h; \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \rightarrow (a|feg|cbd|h; \mathbb{T}_2, \mathbb{T}_1, \mathbb{T}_3; \mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3), \\
\rho &: (a|efg|bcd|h; \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \rightarrow (g|cab|fhe|d; -\mathbb{T}_1, \mathbb{T}_3, -\mathbb{T}_2; \pi - \mathbf{a}_1, \mathbf{a}_3, \pi - \mathbf{a}_2).
\end{aligned} \quad (105)$$

One can show that the weight function (103) is invariant under these substitutions and, thus, possesses the full cube group symmetry.

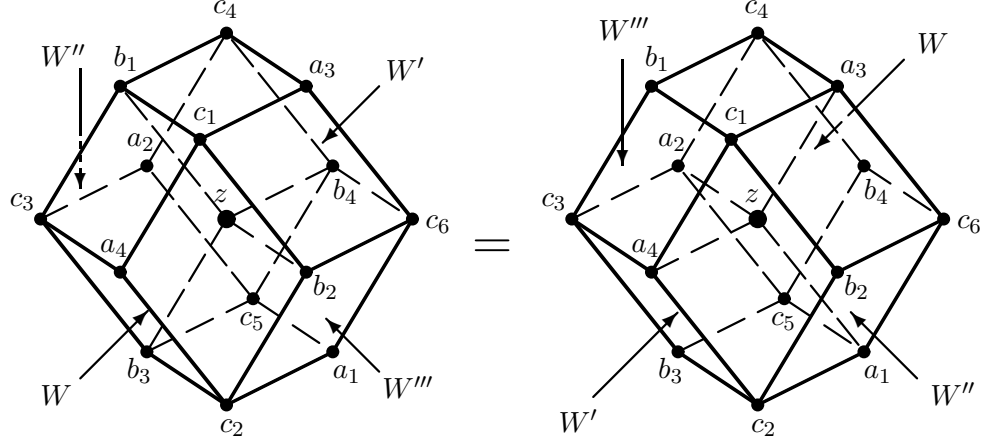


Figure 10: Graphical representation for the tetrahedron equations for interaction-round-a-cube models.

The weight function (103) satisfies the tetrahedron equation of the form [15] (see Fig. 10),

$$\begin{aligned}
& \int_{\mathbb{R}} dz W(a_4|c_2, c_1, c_3|b_1, b_3, b_2|z) W'(c_1|b_2, a_3, b_1|c_4, z, c_6|b_4) \\
& \quad \times W''(b_1|z, c_4, c_3|a_2, b_3, b_4|c_5) W'''(z|b_2, b_4, b_3|c_5, c_2, c_6|a_1) = \\
& \quad = \int_{\mathbb{R}} dz W'''(b_1|c_1, c_4, c_3|a_2, a_4, a_3|z) W''(c_1|b_2, a_3, a_4|z, c_2, c_6|a_1) \\
& \quad \quad \times W'(a_4|c_2, z, c_3|a_2, b_3, a_1|c_5) W(z|a_1, a_3, a_2|c_4, c_5, c_6|b_4),
\end{aligned} \tag{106}$$

where the four sets of the spectral parameters are constrained as

$$T'_2 = T_2, \quad T''_2 = -T_3, \quad T'''_3 = T'_3, \quad T''_1 = T'_1, \quad T'''_2 = T_1, \quad T'''_3 = -T'_1 \tag{107}$$

and

$$a''_1 = a'_1 - a_1, \quad a'''_1 = a'_3 - a_3, \quad a''_2 = a''_2 - a'_2, \quad a'''_3 = a''_3 - a_2. \tag{108}$$

The partition function is defined as

$$Z'_M = \int \cdots \int \prod_{\text{cubes}} W(a|efg|bcd|h) \prod_{\text{vertices}} da \tag{109}$$

where the first product is taken over all elementary cubes and the integral is taken over all vertex spins of the lattice  $a, b, c, \dots$ . This model can be thought of as a non-compact analog of the generalized Zamolodchikov model [16], corresponding to an analytic continuation of the latter to complex values of the number of spin states  $N = -b^2$ .

A precise connection between the two partition functions (83) and (109) requires a detailed discussion of the boundary conditions in both models which is postponed to [33]. Here we only note that expression (83) is particular case of (109) corresponding to a special choice of the field parameters  $a_1, a_2, a_3$  in (103).

## 5 Conclusion

In this paper we have exposed various connections between classical discrete geometry, statistical mechanics and quantum field theory, displaying geometric origins of algebraic structures underlying integrability of quantum systems.

We have shown that the 3D circular lattices are associated with an integrable discrete Hamiltonian system and constructed two different quantizations of this system. The resulting 3D integrable models can be thought of as describing quantum fluctuations of the lattice geometry. The classical geometry of the 3D circular lattices arises as a stationary configuration giving the leading contribution to the partition function of the quantum system in the quasi-classical limit.

We have also obtained two solutions of the tetrahedron equation, which naturally arise in our approach to the quantization of the circular lattices. One of these solutions is new. It has continuous spin variables taking values on the real line. This solution is connected with the modular double of the  $q$ -oscillator algebra and can be considered as a non-compact counterpart of the generalized Zamolodchikov model [16]. The other solution of the tetrahedron equation which arose here in the context of the circular lattices, was previously constructed in [21] by a purely algebraic approach. This solution possesses a remarkable property: it reproduces all two-dimensional solvable models related to finite-dimensional highest weight representations for all quantized affine algebras  $U_q(\widehat{sl}_n)$ ,  $n = 2, 3, \dots, \infty$  (the rank  $n$  coincides with the size of one dimension of the 3D lattice). Plausibly, a similar 3D interpretation, originating from other simple geometrical models, also exists for the trigonometric solutions of the Yang-Baxter equation, related with all other infinite series of quantized affine algebras [42, 43] and superalgebras [44]. Therefore, it might very well be that not only the phenomenon of quantum integrability but the quantized algebras themselves are deeply connected with geometry.

Here we only stated our main results leaving detailed derivations to future publications [33]. There are many other questions we would also like to address there, in particular the geometric meaning of the Poisson algebra (25) and connections of the 3D circular lattices with the 2D circle patterns [11] on the plane or the sphere. It would be interesting to understand underlying reasons of a “persistent” appearance of the  $q$ -oscillator algebra (54) as a primary algebraic structure in many other important aspects of the theory of integrable systems, such as, for example, the construction of Baxter’s  $\mathbf{Q}$ -operators [45] and the calculation of correlation functions of the XXZ model [46]. It would also be interesting to explore connections of our results with the invariants of the 3D manifolds [47–49], the link invariants [50, 51], quantization of the Teichmüller space [52, 53] and the representation theory of  $U_q(sl(2|\mathbb{R}))$  [54]. So, there are many interesting questions about the quantum integrability still remain unanswered, but one thing is getting is more and more clear: it is not just connected with geometry, it is geometry itself! (though the Quantum Geometry).

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