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QUANTIZATION SCHEME FOR MODULAR q-DIFFERENCE EQUATIONS

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We consider modular pairs of certain second-order q-difference equations. An example of such a pair is the t-Q Baxter equations for the quantum relativistic Toda lattice in the strong coupling regime. Another example from quantum mechanics is q-deformation of the Schrödinger equation with a hyperbolic potential. We show that the analyticity condition for the wave function or the Baxter function leads to a set of transcendental equations for the coefficients of the potential or the transfer matrix, the solution of which is their discrete spectrum.

Keywords: Baxter equations, modular dualization, strong coupling regime

1. Introduction

Over the past few years, several models have appeared in the theory of completely integrable quantum systems for which the algebra of observables is a local Weyl algebra with a special value of the Weyl parameter q [1], [2]:

$$q = e^{2\pi i b^2}, \qquad b = e^{i\theta}, \quad 0 < \theta < \frac{\pi}{2}. \tag{1}$$

The key property of this regime for q, known as the strong coupling regime, is that the complex conjugation of q is equivalent to a modular Jacobi transformation, $q^* = e^{-2\pi i b^{-2}}$ (we use the asterisk to indicate complex conjugation).

We assume that the coordinate and momentum operators $\mathbf{x}$ and $\mathbf{p}$ of the Heisenberg algebra,

$$\mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \frac{i}{2\pi}, \tag{2}$$

are used to construct the Weyl pair $\mathbf{u}$, $\mathbf{v}$,

$$\mathbf{u} = e^{2\pi b\mathbf{x}}, \qquad \mathbf{v} = e^{2\pi b\mathbf{p}}, \qquad \mathbf{u}\mathbf{v} = q\mathbf{v}\mathbf{u}, \tag{3}$$

and the modular dual (i.e., conjugate) pair

$$\mathbf{u}^\dagger = e^{2\pi b^{-1}\mathbf{x}}, \qquad \mathbf{v}^\dagger = e^{2\pi b^{-1}\mathbf{p}}, \qquad \mathbf{v}^\dagger\mathbf{u}^\dagger = q^*\mathbf{u}^\dagger\mathbf{v}^\dagger. \tag{4}$$

It is obvious that any element of the first pair commutes with any element of the second pair. The physical interpretation of regime (1) is that *any polynomial* $\mathbf{J}(\mathbf{u},\mathbf{v})$ commutes with its dual,

$$\mathbf{J}(\mathbf{u},\mathbf{v})\mathbf{J}(\mathbf{u},\mathbf{v})^\dagger = \mathbf{J}(\mathbf{u},\mathbf{v})^\dagger\mathbf{J}(\mathbf{u},\mathbf{v}), \tag{5}$$

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i.e., we can speak of commuting Hermitian operators $\mathbf{J}+\mathbf{J}^\dagger$ and $i(\mathbf{J}-\mathbf{J}^\dagger)$ related to physical Hamiltonians. The spectral problem is the pair of equations for the vector $\langle\Psi|$:

$$\langle\Psi|\cdot\mathbf{J}(\mathbf{u},\mathbf{v}) = \langle\Psi|\cdot\mathbf{J}(\mathbf{u},\mathbf{v})^\dagger = 0. \tag{6}$$

These equations are sometimes called the equations on the quantum curve (because the equation $J(u,v)=0$ as $q\to 1$ determines a classic algebraic curve). Equations (6) are also the basis-invariant form of the t-Q Baxter equations, $\langle\Psi|\equiv\langle Q|$. In terms of quantum mechanics, the polynomials $\mathbf{J}$ and $\mathbf{J}^\dagger$ can be understood as deformations of the Schrödinger operator $\mathbf{H}-E$.

The problem with difference equations (6) is that the proper asymptotic condition for Ψ in, for example, the coordinate representation (the analogue of the quantum mechanical condition $\Psi(x)\in L_2(\mathbb{R})$) does not determine Ψ uniquely and does not lead to a quantization of the spectrum.

In this paper, we show that if the proper asymptotic condition is supplemented by the analyticity condition for the wave function at a certain pole, then the wave function is determined uniquely and the spectrum is quantized. The spectrum quantization equations are a system of rather intricate equations in terms of certain transcendental, but well-defined, functions on the coefficients of the polynomials $\mathbf{J}$ and $\mathbf{J}^\dagger$. By virtue of transcendence, this finite system has an infinite number of solutions.

We begin our exposition with the quantum mechanical approach, in terms of which the analyticity condition is natural, and we construct the wave function and the spectrum quantization conditions step by step. We discuss the Baxter equation for the modular Toda lattice in the penultimate section.

2. The quantum mechanics of a q-hyperelliptic pendulum

We consider an operator-valued polynomial $\mathbf{J}(\mathbf{u},\mathbf{v})$ of the special form

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \mathbf{v}+\mathbf{v}^{-1}+T(\mathbf{u}), \tag{7}$$

where the potential (or rather the transfer matrix) has the form

$$T(u) = \lambda u^{-L}\sum_{j=0}^{2L} t_j\cdot(-u)^j, \qquad t_0 = t_{2L} = 1, \tag{8}$$

and all the t_j and λ are complex coefficients. This polynomial $\mathbf{J}$ can be seen as a q-deformation of the Schrödinger operator $\mathbf{j}=\mathbf{p}^2+V(\mathbf{x})-E$ with a hyperbolic potential. We note that as $q\to 1$, the equation $J(u,v)=0$ determines a hyperelliptic curve of genus $g=2L-1$ with g modules t_j, and the dynamics of u and v with the bracket $\{u,v\}=uv$ are those of a hyperelliptic pendulum from the mechanical standpoint.

We use the coordinate representation,

$$\langle\Psi|x\rangle = \Psi(x), \qquad \langle\Psi|\mathbf{x}|x\rangle = x\Psi(x), \qquad \langle\Psi|\mathbf{p}|x\rangle = \frac{i}{2\pi}\Psi'(x). \tag{9}$$

It follows from (3) and (4) that

$$\begin{aligned} \langle\Psi|\mathbf{u}|x\rangle &= e^{2\pi xb}\Psi(x), \qquad \langle\Psi|\mathbf{u}^\dagger|x\rangle = e^{2\pi xb^{-1}}\Psi(x),\\ \langle\Psi|\mathbf{v}|x\rangle &= \Psi(x+ib), \qquad \langle\Psi|\mathbf{v}^\dagger|x\rangle = \Psi(x+ib^{-1}). \end{aligned} \tag{10}$$

For brevity in the following formulas, we set

$$u\equiv e^{2\pi xb}, \qquad \tilde u\equiv e^{2\pi xb^{-1}} \tag{11}$$

for every $x \in \mathbb{C}$. We note that if x is complex, then $\tilde{u} \neq u^*$.

We now turn to Eqs. (6) in the coordinate representation. For polynomial (7), they have the form

$$\begin{aligned} &\Psi(x+ib) + \Psi(x-ib) + \Psi(x)T(u) = 0, \\ &\Psi(x+ib^{-1}) + \Psi(x-ib^{-1}) + \Psi(x)T^*(\tilde{u}) = 0, \end{aligned} \tag{12}$$

where T^* indicates complex conjugation of all coefficients of the Laurent series of T. The function $\Psi(x)$ for $x \in \mathbb{R}$ is the wave function, and the first condition on Ψ is therefore trivial: $\Psi(x) \in L_2(\mathbb{R})$. It is clear that the shift of the argument of Ψ to the complex domain, which appears in the second line in representation (10), must be seen as the analytic continuation of the physical function $\Psi(x)\big|_{x\in\mathbb{R}}$. In terms of quantum mechanics, a complex argument shift is a series expansion (we recall that $ib = i\cos\theta - \sin\theta$):

$$\Psi(x+ib) = \sum_{n=0}^{\infty} \Psi^{(n)}(x - \sin\theta)\frac{(i\cos\theta)^n}{n!}, \tag{13}$$

which is well defined if and only if $\Psi(x)$ is analytic in the band

$$-\cos\theta < \operatorname{Im} x < \cos\theta. \tag{14}$$

3. Holomorphic parts

With a slight modification, every equation in system (12) can be written as an equation having a solution holomorphic with respect to $u^{\pm 1}$ or $\tilde{u}^{\pm 1}$. We introduce the notation

$$t_+(u) = \frac{u^L}{\lambda}T(u) = \sum_{j=0}^{2L} t_j \cdot (-u)^j, \qquad t_-(u) = \frac{1}{u^L\lambda}T(u) = \sum_{j=0}^{2L} t_{2L-j} \cdot (-u)^{-j} \tag{15}$$

and consider the two equations

$$\chi_+(q^{-1}u) = t_+(u)\chi_+(u) - Gu^{2L}\chi_+(qu), \tag{16}$$

$$\chi_-(qu) = t_-(u)\chi_-(u) - Gu^{-2L}\chi_-(q^{-1}u), \tag{17}$$

where $G \equiv q^L\lambda^{-2}$.

We suppose that $\chi_+(u)$ is a solution of Eq. (16) holomorphic with respect to u such that $\chi_+(0) = 1$. We suppose that $\chi_-(u)$ is a solution of Eq. (17) holomorphic with respect to u^{-1} such that $\chi_-(\infty) = 1$. Both these functions are uniquely defined and exist for any t_j and G. We substitute the expansion

$$\chi_+(u) = \sum_{n=0}^{\infty} \chi_n \cdot (-u)^n \tag{18}$$

in (16), for example, and consider the coefficient of u^k. The condition that this coefficient vanishes is equivalent to the condition that $(q^{-k} - 1)\chi_k$ is a linear combination of the quantities χ_n for $0 \leq n < k$, the coefficients t_+ and G. This is obviously a simple recursion, determined uniquely by the initial value $\chi_0 = 1$. Moreover, we can also obtain an upper bound $|\chi_n| < |q|^{n^2/4L}X^n$, where the inessential quantity X is determined by the set t_j and G.

Furthermore, we can represent χ_+ as a semi-infinite matrix product. Let

$$L(u) = \begin{pmatrix} t_+(u) & -Gu^{2L} \\ 1 & 0 \end{pmatrix} \tag{19}$$

and

$$L_n(u) = L(u)L(qu)(q^2u)\cdots L(q^{n-1}u). \tag{20}$$

For $|q| < 1$, the matrix product $L_n(u)$ converges absolutely, and

$$L_\infty(u) = \begin{pmatrix} \chi_+(q^{-1}u) & 0 \\ \chi_+(u) & 0 \end{pmatrix}. \tag{21}$$

This equality follows from the relation $L_\infty(u) = L(u)L_\infty(qu)$.

We can write similar expressions for χ_-. For example, if $t_-(u) = t_+(u^{-1})$ (i.e., $T(u) = T(u^{-1})$), then $\chi_-(u) \equiv \chi_+(u^{-1})$.

The semiclassical limits of $\chi_\pm(u)$ are remarkable. The function $\psi(u) = \prod_{n=0}^{\infty}(1 - q^n u)$ is known in certain applications as the compact quantum dilogarithm because it has the asymptotic behavior

$$\psi(u) = \exp\biggl\{\epsilon^{-1}\int_0^u \log(1-x)\,d\log x + O(1)\biggr\}$$

in the limit $q = e^{-\epsilon} \to 1$. In the same limit, the holomorphic solution

$$\chi(q^{-1}u) = t(u)\chi(u) - p(u)\chi(qu), \tag{22}$$

where the polynomials $t(u)$ and $p(u)$ are such that $t(0) = 1$ and $p(0) = 0$ (see (16)), has the asymptotic behavior

$$\chi(u) = \exp\biggl\{\epsilon^{-1}\int_0^u \log y\,d\log x + O(1)\biggr\}, \tag{23}$$

where (x, y) is a point on the hyperelliptic curve $y + p(x)y^{-1} = t(x)$. The initial point in integral (23) is $(x, y) = (0, 1)$, and we integrate in the neighborhood of this point.

4. The wave function ansatz

With notation (11) taken into account, the functions

$$\psi_+(x) = \frac{\chi_+(u)}{n_+(u)}, \qquad \psi_-(x) = \frac{\chi_-(u)}{n_-(u)} \tag{24}$$

are solutions of the first equation in (12) if

$$\frac{n_+(u)}{n_+(q^{-1}u)} = -\lambda u^{-L}, \qquad \frac{n_-(u)}{n_-(qu)} = -\lambda u^{L}. \tag{25}$$

The solutions of the second equation in (12) are

$$\psi_+^*(x) = \frac{\chi_+^*(\tilde u)}{n_+^*(\tilde u)}, \qquad \psi_-^*(x) = \frac{\chi_-^*(\tilde u)}{n_-^*(\tilde u)}. \tag{26}$$

Equations (25) determine a class of functions n_+ and a class of functions n_- (i.e., the $n_\pm$ are determined up to doubly periodic functions); these classes are not yet fixed. The general solution of Eqs. (12) is a linear combination of four terms:

$$\Psi(x) \sim \sum_{\varepsilon,\varepsilon'=\pm} \psi_\varepsilon(u)\psi^*_{\varepsilon'}(\tilde{u}). \tag{27}$$

We can approximate the average asymptotic behavior of each term. We suppose that $f(x)$ has the following asymptotic difference properties as $x \to \pm\infty$:

$$\frac{f(x)}{f(x+ib)} \sim a e^{2\pi x b\varepsilon}, \qquad \frac{f(x)}{f(x-ib^{-1})} \sim a' e^{2\pi x b^{-1}\varepsilon'}, \tag{28}$$

where the ε and ε' are integers. The argument shifts in the difference equations can be combined in a real argument shift $x \mapsto x + i(b - b^{-1})$, and the asymptotic behavior of $|f(x)|$ with real $x \to \pm\infty$ can be easily approximated as

$$\big|f(x)\big| \le F(x)\exp\biggl(\frac{\pi}{2}(\varepsilon+\varepsilon')\coth\theta x^2 + \pi(\varepsilon+\varepsilon')\cos\theta x + \frac{\log|aa'|}{2\sin\theta}x\biggr), \tag{29}$$

where $F(x)$ is a certain periodic function with the period $i(b - b^{-1}) = -2\sin\theta$. Averaging the asymptotic expression implies substituting $F(x) \mapsto \max F(x) = \text{const}$. Therefore, we conclude that the $\psi_\pm(u)\psi^*_\pm(\tilde{u})$ increase as $x \to \pm\infty$, whereas

$$\big|\psi_\pm(u)\psi^*_\mp(\tilde{u})\big| \sim e^{-2\pi L\cos\theta|x|}, \quad x \to \pm\infty. \tag{30}$$

Therefore, we obtain the ansatz

$$\Psi(x) = \frac{\chi_+(u)\chi^*_-(\tilde{u})}{D_1(x)} - \frac{\chi_-(u)\chi^*_+(\tilde{u})}{D_2(x)} \tag{31}$$

for $\Psi(x) \in L_2(\mathbb{R})$, where the $D_1 \sim n_+ n^*_-$ and $D_2 \sim n_- n^*_+$ satisfy the conditions

$$\begin{aligned} \frac{D_1(x)}{D_1(x-ib)} &= -\lambda u^{-L}, \qquad & \frac{D_1(x)}{D_1(x-ib^{-1})} &= -\lambda^* \tilde{u}^L, \\ \frac{D_2(x)}{D_2(x+ib)} &= -\lambda u^{L}, \qquad & \frac{D_2(x)}{D_2(x+ib^{-1})} &= -\lambda^* \tilde{u}^{-L}. \end{aligned} \tag{32}$$

As before, these equations define classes of functions. We further suppose that

$$-\lambda = e^{i\pi L b^2 - \pi\gamma b}, \qquad G = e^{2\pi\gamma b}, \tag{33}$$

where $\gamma \in \mathbb{R}$. Then D_1 and D_2 can be written in the form

$$D_1(x) = e^{-i\pi L x^2 + i\pi\gamma} u^{-L} W_1(u), \qquad D_2(x) = e^{-i\pi L x^2 - i\pi\gamma} u^{-L} W_2(u), \tag{34}$$

where the functions

$$W_{1,2}(u) = u^{2L} W_{1,2}(qu) \tag{35}$$

are $(x \mapsto x - ib^{-1})$-periodic parts of $D_{1,2}$. As before, the functions W_1 and W_2 are determined by these equations up to doubly periodic functions. According to the Liouville theorem, Eqs. (35) imply that $W_{1,2}(u)$, as functions of x, have at least $2L$ zeros in the parallelogram of periods. Of course, the zeros can be chosen to be complex. Therefore, $\Psi(x) \in L_2(\mathbb{R})$, and the spectrum t_j is arbitrary. For the quantization, we turn to the analyticity condition for $\Psi(x)$ in band (14). This band contains the entire parallelogram of periods; therefore, there is only one method to guarantee the analyticity of $\Psi(x)$ in the band. This method presumes the following:

1. We must choose the function W_2 to be proportional to W_1, both the summands in (31) having a common denominator and

$$\Psi(x) = e^{i\pi Lx^2} \frac{e^{-i\pi\gamma x}\chi_+(u)\chi_-^*(\tilde{u}) - \xi e^{i\pi\gamma x}\chi_-(u)\chi_+^*(\tilde{u})}{u^{-L}W(u)}, \tag{36}$$

   where ξ is a complex number, $W_1 = W_2 = W$.

2. We must choose the function $W(u)$ with the smallest possible set of $2L$ zeros,

$$W(u) = \prod_{j=1}^{2L} H(u/s_j), \tag{37}$$

   where $H(u) = -uH(qu)$ is the theta function

$$H(u) = (u;q)_\infty (qu^{-1};q)_\infty (q;q)_\infty = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2}(-u)^n, \tag{38}$$

   and Eqs. (35) then imply the restriction

$$\prod_{j=1}^{2L} s_j = 1. \tag{39}$$

3. We must guarantee that the zeros of the denominator cancel the zeros of the numerator in formula (36) in the entire band. (This method was used by Gutzwiller in [3].)

The last condition generates an infinite system of equations for a set of $2L-1$ independent variables s_j, $j = 1, \ldots, 2L-1$, and $2L-1$ parameters t_j, $j = 1, \ldots, 2L-1$. Fortunately, there is a special choice of s_j that allows reducing the infinite system to a finite one. We perform this reduction in the next section.

5. Quantization equations

In addition to notation (11) and (33), it is also convenient to introduce the quantities

$$s_j = e^{2\pi\sigma_j b}, \qquad \sum_{j=1}^{2L} \sigma_j = 0. \tag{40}$$

The zeros of the denominator in equality (36) in band (14) are the points $\sigma_j + inb - inb^{-1}$, $j = 1, \ldots, 2L$, $n \in \mathbb{Z}$, with the corresponding $u = q^n s_j$ and $\tilde{u} = q^{*n}\tilde{s}_j$. The vanishing condition for the numerator in (36) at these points gives an infinite set of equations

$$\xi = e^{-2\pi i\gamma\sigma_j} \frac{G^n \dfrac{\chi_+(q^n s_j)}{\chi_-(q^n s_j)}}{G^{*n} \dfrac{\chi_+^*(q^{*n}\tilde{s}_j)}{\chi_-^*(q^{*n}\tilde{s}_j)}}, \quad n \in \mathbb{Z}, \quad j = 1, \ldots, 2L. \tag{41}$$

The numerator and denominator in each of these equations are conjugate, and a more detailed analysis shows that all these equations can be satisfied if both the numerator and the denominator are independent of n, i.e.,

$$G^n \frac{\chi_+(q^n s_j)}{\chi_-(q^n s_j)} = \frac{\chi_+(s_j)}{\chi_-(s_j)} \overset{\text{def}}{=} M(s_j). \tag{42}$$

The condition of independence from n can be written in the following form. We suppose that

$$w(u) \overset{\text{def}}{=} \chi_+(q^{-1}u)\chi_-(u) - G\chi_+(u)\chi_-(q^{-1}u). \tag{43}$$

Then condition (42) can be written as $w(q^n s_j) = 0$ for any n and j.

The function w is related to the q-Wronskian $\psi_+(q^{-1}u)\psi_-(u) - \psi_+(u)\psi_-(q^{-1}u)$ in Eq. (24). Excluding $n_\pm$, we can obtain equality (43). From Eqs. (16) and (17) for $\chi_\pm$, we obtain $w(u) = u^{2L}w(qu)$, and because series (43) converges in u and u^{-1} by definition, there exists a unique set s_j, $j = 1, \dots, 2L$, such that

$$w(u) = C\prod_{j=1}^{2L} H(u/s_j), \qquad \prod_{j=1}^{2L} s_j = 1. \tag{44}$$

The constant C, depending on G and on the set t_j, is inessential, and $W(u) = w(u)/C$.

Therefore, if the s_j are zeros of Wronskian (43), then the entire infinite set of Eqs. (41) reduces to a finite set of $2L-1$ equations (because the number ξ itself must be determined by these equations)

$$\xi = e^{-2\pi i\gamma\sigma_j} \frac{M(s_j)}{M^*(\tilde{s}_j)}, \quad j = 1, \dots, 2L. \tag{45}$$

This is a system of equations for the coefficients t_j of potential (8), the number of equations coinciding with the number of unknowns t_j and the number of independent quantities s_j. It follows from formulas (43) and (44) that s_j is a single-valued function of the set t_j. On the other hand, it can be shown that the t_j are multivalued functions of s_j with an infinite number of sheets, which guarantees that the number of solutions of Eq. (45) is infinite.

Returning to the end of the preceding section, we can verify that condition (37) is automatically satisfied: it is not a hypothesis but follows from (41) and the subsequent analysis.

Numerical analysis of equations (45) allows supposing that all their solutions correspond to real σ_j, and ξ is hence a pure phase, the $\Psi(x)$ from (36) having a constant phase if $x \in \mathbb{R}$ (because complex conjugation is equivalent to a modular transformation). Moreover, if the potential is symmetric, $T(u) = T(u^{-1})$, then $\xi = \pm 1$ is simply a parity.

In conclusion, we note that by virtue of definition (42), the numerator in (36) is equal to zero at the points $\sigma_j + inb - imb^{-1}$, $n, m \in \mathbb{Z}$, and $\Psi(x)$ is therefore analytic in the entire complex plane.

6. The quantum relativistic Toda lattice

The quantum curve for the quantum relativistic Toda lattice of length N in one of the possible normalizations [4] is

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \mathbf{v} + (-)^N G\mathbf{u}^N\mathbf{v}^{-1} - t(\mathbf{u}), \tag{46}$$

where

$$t(u) = \sum_{j=0}^{N} t_j(-u)^j, \qquad t_0 = t_N = 1. \tag{47}$$

The Baxter equations for $Q(x) = \langle Q|x\rangle$ are $\langle Q|\mathbf{J} = \langle Q|\mathbf{J}^\dagger = 0$.

The conditions on $Q(x)$ are [2], [5]

1. $\bigl|Q(x)\bigr| \sim 1$ as $x \to -\infty$, $\bigl|Q(x)\bigr| \sim e^{-2\pi N\cos\theta x}$ as $x \to +\infty$, and
2. $Q(x)$ is an entire function.

Slightly modifying the previous arguments (they correspond to the case of even $N = 2L$), we define $\chi_\pm$ as solutions holomorphic with respect to $u^{\pm 1}$:

$$\begin{aligned}\chi_+(q^{-1}u) &= t(u)\chi_+(u) - G(-u)^N\chi_+(qu),\\ \chi_-(qu) &= \frac{t(u)}{(-u)^N}\chi_-(u) - \frac{G}{(-u)^N}\chi_-(q^{-1}u).\end{aligned} \tag{48}$$

Their q-Wronskian is determined by formula (43). We also have the difference relation $W(u) = (-u)^N W(qu)$ and the expansion

$$W(u) = C\prod_{j=1}^{N} H(u/s_j), \qquad \prod_{j=1}^{N} s_j = 1. \tag{49}$$

As in the preceding sections, we suppose that

$$M(s_j) = \frac{\chi_+(s_j)}{\chi_-(s_j)}, \tag{50}$$

and the quantization conditions for the real σ_j, $s_j \equiv e^{2\pi\sigma_j b}$, are

$$e^{-2\pi i\gamma\sigma_j}\frac{M(s_j)}{M(s_j)^*} = \xi, \quad j = 1, \dots, N. \tag{51}$$

All notation is the same as used above. Then the function $Q(x)$ satisfying all conditions is given by the formula

$$Q(x) = \frac{e^{-2\pi i\gamma x}\chi_+(u)\chi_-^*(\tilde{u}) - \xi\chi_-(u)\chi_+^*(\tilde{u})}{W(u)}. \tag{52}$$

7. Conclusion

We propose a quantization scenario for certain Baxter-type equations in the strong coupling regime. The first important point of the scenario is that certain second-order q-difference equations have solutions holomorphic with respect to u (the functions χ), just as the compact quantum dilogarithm is a holomorphic solution of a first-order difference equation [6]. These functions are uniquely defined and are perfect for numerical analysis, which distinguishes our case from the case $|q| = 1$ [5]. Another important observation is that the q-Wronskian of two functions χ is a product of theta functions, and the Wronskian must be used to transform the functions χ to solutions of the initial difference equations. Using the zeros of the Wronskian reduces an infinite system for the residues to a finite system.

The quantization condition, i.e., the analyticity condition for the wave function or the Baxter function Q in band (14), can be written as a set of transcendental equations (45) in terms of the functions χ. The number of equations in (45) increases as the system size increases. We have not yet been able to find an adequate method for studying these equations in the thermodynamic limit.

In this paper, we considered specific quantum curves $\mathbf{J}(\mathbf{u},\mathbf{v})$ of hyperelliptic type (Eqs. (7) and (46)). In both examples, $\Psi(x)$ and $Q(x)$ turned out to be analytic functions on the entire complex plane; this is a peculiarity of $\mathbf{J}(\mathbf{u},\mathbf{v})$ of the Toda type. This is not necessarily so in more complex examples. The proposed scheme can be generalized to the class of all quantum algebraic varieties (not just hyperelliptic ones) corresponding to a class of integrable systems with a local Weyl algebra as the algebra of observables [7].

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