

**INTRODUCTION TO LIE ALGEBRAS
AND THEIR REPRESENTATION THEORY
MATH3349**

V. MANGAZEEV AND S. SERGEEV

ABSTRACT. A draft for a course of lectures delivered in first semester of 2006.

Purpose of the first part:

- Structure theory: classification of all possible simple (semi-simple) Lie **finite dimensional** algebras
- Representation theory: for a given simple Lie algebra classify all its **finite-dimensional** representations.

Our heroes:

- Niels Henrik Abel, 1802–1829 [Relation to Lie algebras: commutative algebra is called the Abelian algebra. In this case it is not an algebra, it is an arithmetics].
- Marius Sophus Lie, 1842–1899
- Élie Joseph Cartan, 1869–1951
- Hermann Weyl, 1885–1955

Purpose of the second part:

- Modern development of Lie algebras: infinite algebras, quantum groups, etc.

This text is the subject of permanent improvement.

Experimental fact: an alive lecture equals to three full pages plus one paragraph on fourth page.

1. DEFINITIONS

1.1. Generic algebra.

Definition 1. Algebra \mathfrak{A} is a linear space over field \mathbb{F} with given distributive bilinear composition operation φ .

Comments. \mathfrak{A} is the linear space, i.e.

$$(1) \quad \forall x, y \in \mathfrak{A} \quad \text{and} \quad \forall a, b \in \mathbb{F} \quad ax + by \in \mathfrak{A} .$$

Composition operation $\varphi : \mathfrak{A} \times \mathfrak{A} \mapsto \mathfrak{A}$:

$$(2) \quad \forall x, y \in \mathfrak{A} \quad \varphi(x, y) \in \mathfrak{A} .$$

Distributivity condition:

$$(3) \quad \forall x, y, z \in \mathfrak{A} \quad \varphi(x, y + z) = \varphi(x, y) + \varphi(x, z) , \quad \varphi(x + y, z) = \varphi(x, z) + \varphi(y, z)$$

Composition is bilinear:

$$(4) \quad \forall x, y \in \mathfrak{A} \quad \text{and} \quad \forall a, b \in \mathbb{F} \quad \varphi(ax, by) = ab\varphi(x, y) .$$

Field \mathbb{F} : we will mostly use the fields \mathbb{R} and \mathbb{C} .

Definition 2. A subspace \mathfrak{B} of an algebra \mathfrak{A} is a subalgebra if it is closed under the composition (notation $\mathfrak{B} \subseteq \mathfrak{A}$).

Since the algebra is the linear space, one may define its basis:

$$(5) \quad \{x_\alpha \in \mathfrak{A} | \alpha \in I\} \quad : \quad \forall x \in \mathfrak{A} \quad x = \sum_{\alpha \in I} \zeta_\alpha x_\alpha , \quad \zeta_\alpha \in \mathbb{F} .$$

For a given basis, the algebra is defined by its structure constants $c_{\alpha, \beta}^\gamma$:

$$(6) \quad \varphi(x_\alpha, x_\beta) = \sum_{\gamma \in I} c_{\alpha, \beta}^\gamma x_\gamma$$

1.2. Associative algebra.

Definition 3. Algebra \mathfrak{A} is called associative if the composition operation is associative

$$(7) \quad \forall x, y, z \in \mathfrak{A} \quad \varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$$

The composition operation for the associative algebras is denoted (and usually called) as multiplication

$$(8) \quad \varphi(x, y) = xy$$

Definition 4. $gl(n, \mathbb{F})$ is the set of matrices of the size $n \times n$ with the entries in \mathbb{F} .

Let V be a vector space over field \mathbb{F} , $n = \dim V$. The set $gl(n, \mathbb{F})$ is the set of linear transformations (endomorphisms) of V , and therefore $gl(n, \mathbb{F})$ is called sometimes $\text{End}(V)$.

$gl(n, \mathbb{F})$ is a linear space, the matrix product is associative, therefore

Statement 1. $gl(n, \mathbb{F})$ with composition defined as the matrix multiplication is the associative algebra.

1.3. Lie algebra.

Definition 5. An algebra with composition rule denoted by $\varphi(x, y) = [xy]$ satisfying

$$(9) \quad [xy] = -[yx], \quad [[xy]z] + [[yz]x] + [[zx]y] = 0$$

is called a Lie algebra.

The last relation in (9) is called the Jacobi identity. Composition $[xy]$ is called the Lie brackets. Sometimes it is convenient to write the comma, $[x, y]$.

Statement 2. $gl(n, \mathbb{F})$ with brackets defined as the commutator,

$$(10) \quad [xy] = xy - yx$$

is the Lie algebra

To verify the statement, it is enough to check that commutator satisfies the Jacobi identity (9).

Definition 6. *The set of elements $Z(\mathfrak{A}) = \{ z \mid [zx] = 0 \ \forall x \in \mathfrak{A} \}$ is called the centre of Lie algebra \mathfrak{A}*

The Lie algebra $gl(n, \mathbb{F})$ has a one-dimensional centre spanned by the unity matrix.

Statement 3. *The space of smooth real functions of two real variables x and p is the Lie algebra over \mathbb{R} with the bracket*

$$(11) \quad [fg] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} .$$

Actually, this bracket is called the Poisson bracket and denoted by $\{f, g\}$, and the algebra is called Poisson-Lie algebra.

Definition 7. *A linear transformation $\pi : \mathfrak{A} \mapsto \mathfrak{A}'$ is an algebra homomorphism if it preserves the operations which define the algebras.*

Comments. π is the transformation, i.e.

$$(12) \quad \forall x \in \mathfrak{A} \quad \pi(x) \in \mathfrak{A}' .$$

π is the linear transformation,

$$(13) \quad \forall x, y \in \mathfrak{A} \quad \text{and} \quad \forall a, b \in \mathbb{F} \quad \pi(ax + by) = a\pi(x) + b\pi(y) .$$

It preserves the structure:

$$(14) \quad \pi([x, y]) = [\pi(x), \pi(y)] .$$

Definition 8. *A representation of a Lie algebra \mathfrak{A} is a homomorphism $\pi : \mathfrak{A} \mapsto \mathfrak{B}$ to an algebra $\mathfrak{B} \subseteq \text{End}(V)$ of the endomorphisms of some vector space V . If the map is one-to-one, the representation is said to be faithful.*

2. ALGEBRA $\mathfrak{o}(3, \mathbb{R})$

Lie algebras arise naturally as tangent spaces of Lie groups.

Definition 9. A set G is called a group if

- it is defined the product on G : $g_1g_2 \in G, \forall g_1, g_2 \in G$
- there exists the unity element 1 : $g1 = 1g = g, \forall g \in G$
- and any element has its inverse: $\forall g \in G \exists g^{-1} \in G: gg^{-1} = g^{-1}g = 1$

If G is in addition a smooth variety, it is called the Lie group.

The basic example is the group of rotations of Euclidean space \mathbb{R}^3 . Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the orthogonal basis of \mathbb{R}^3 (frame of reference) with the scalar products

$$(15) \quad (\mathbf{e}_j, \mathbf{e}_k) = \delta_{j,k} .$$

Any vector has the unique decomposition

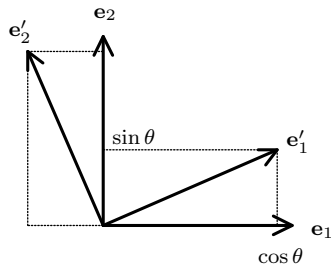
$$(16) \quad \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 .$$

Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be another orthogonal frame of reference of \mathbb{R}^3 obtained by a *smooth* turn of the initial frame. According to (16), one may express \mathbf{e}'_j in the terms of \mathbf{e}_j :

$$(17) \quad \mathbf{e}'_j = \sum_{k=1,2,3} u_{j,k}\mathbf{e}_k \quad \text{or, in matrix form,} \quad \mathbf{e}' = u \cdot \mathbf{e} .$$

The group structure is the superposition of transformations: if basis \mathbf{e}'' is decomposed with respect to \mathbf{e}' , $\mathbf{e}'' = u_2 \cdot \mathbf{e}'$, and \mathbf{e}' is decomposed with respect to \mathbf{e} , $\mathbf{e}' = u_1 \cdot \mathbf{e}$, then $\mathbf{e}'' = u \cdot \mathbf{e}$, $u = u_2 \cdot u_1$.

The basic tool for the geometrical description of transformations is a rotation in two-dimensional subspace. For the turn of the frame $(\mathbf{e}_1, \mathbf{e}_2)$ by the angle θ the Pythagoras theorem reads



$$\begin{aligned} \mathbf{e}'_1 &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \\ \mathbf{e}'_2 &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 \end{aligned}$$

In the matrix form

$$(18) \quad \mathbf{e}' = u(\theta) \cdot \mathbf{e}, \quad u(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \equiv e^{\theta T} \quad \text{where} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exponent of a matrix is defined by the series expansion

$$(19) \quad e^A \stackrel{\text{def}}{=} 1 + A + \frac{1}{2}A^2 + \dots + \frac{1}{n!}A^n + \dots$$

and in the case of (18) the key relation is $T^2 = -1$.

There exists a particular parameterization of the element u (17) of orthogonal three-dimensional group in the terms of Euler angles¹ corresponding to a decomposition of u into simple rotations (18):

$$(20) \quad u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix may be rewritten in exponential form,

$$(21) \quad u = e^{\psi T_1} \cdot e^{\theta T_2} \cdot e^{\phi T_3}$$

where

$$(22) \quad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

¹We want to connect basis \mathbf{e}' and basis \mathbf{e} of three dimensional Euclidean space. It may be done in three steps:

- (1) Suppose, the planes $(\mathbf{e}_1\mathbf{e}_2) \perp \mathbf{e}_3$ and $(\mathbf{e}'_2\mathbf{e}'_3) \perp \mathbf{e}'_1$ do not coincide. Then there exists a turn around \mathbf{e}_3 putting the image of \mathbf{e}_2 into the plane $(\mathbf{e}'_2\mathbf{e}'_3)$. Denote the result of the turn as $\mathbf{e}^{(1)}$, $\mathbf{e}^{(1)} = u_3(\phi) \cdot \mathbf{e}$.
- (2) As the result, $\mathbf{e}_2^{(1)} \perp \mathbf{e}'_1, \mathbf{e}_2^{(1)}, \mathbf{e}_3^{(1)}$, i.e. the last three vectors are complanar. Therefore, there exists a turn around $\mathbf{e}_2^{(1)}$ putting the image of $\mathbf{e}_1^{(1)}$ to \mathbf{e}'_1 . Denote the result as $\mathbf{e}^{(2)}$: $\mathbf{e}^{(2)} = u_2(\theta) \cdot \mathbf{e}^{(1)}$.
- (3) Now we have $\mathbf{e}'_1 = \mathbf{e}_1^{(2)} \perp \mathbf{e}_2^{(2)}, \mathbf{e}_3^{(2)}, \mathbf{e}'_2, \mathbf{e}'_3$. Therefore there exists a turn putting the images of $\mathbf{e}_2^{(2)}$ and $\mathbf{e}_2^{(3)}$ exactly into \mathbf{e}'_2 and \mathbf{e}'_3 : $\mathbf{e}' = u_1(\psi) \cdot \mathbf{e}^{(2)}$.

Therefore, $\mathbf{e}' = u_1(\psi) \cdot u_2(\theta) \cdot u_3(\phi) \cdot \mathbf{e}$, exactly the decomposition of (20).

Now let us test the exponential decomposition from the group point of view. Let two elements of a group are given in the exponential form, $g_1 = e^A$ and $g_2 = e^B$. Their product

$$(23) \quad e^A \cdot e^B = e^{C(A,B)} \quad \text{where} \\ C(A,B) \equiv A + B + \frac{1}{2}[A, B] + \frac{1}{12} \left([[A, B], B] + [A, [A, B]] \right) - \frac{1}{720} \dots$$

must be as well the element of the group. Expression for $C(A, B)$ is a well defined infinite series. Formula (23) is the *identity* known as the Baker-Campbell-Hausdorff identity². The key point is that all non-linear terms in the series (23) are the matrix commutators. Therefore, e^C belongs to the same class as e^A and e^B if and only if A and B belong to a Lie algebra.

The elements T_j (22) do form a linear space closed under the commutator operation:

$$(25) \quad [T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2.$$

Definition 10. *The Lie algebra (25) is called $o(3, \mathbb{R})$.*

Sometimes it is convenient to insert the imaginary unity to the definition of T : $L_j \stackrel{\text{def}}{=} iT_j$. Since T_j are skew-symmetric matrices, L_j are hermitian ones. It follows from (25),

$$(26) \quad [L_1, L_2] = iL_3, \quad [L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2.$$

At this point it is well to recollect a bit the classical and quantum mechanics. The rotational symmetry in mechanics provides the conservation of angular momentum \vec{L} ,

$$(27) \quad L_1 = x_2p_3 - x_3p_2, \quad L_2 = x_3p_1 - x_1p_3, \quad L_3 = x_1p_2 - x_2p_1$$

In quantum mechanics the coordinate and momentum are operators of the Heisenberg algebra

$$(28) \quad [x_j, p_k] = i\delta_{j,k}.$$

The components of the angular momentum become the operators as well. Their algebra is given by (26).

²More famous BCH-identity is

$$(24) \quad e^A \cdot B \cdot e^{-A} = B + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \dots [A, B] \dots]]}_{n \text{ times}}$$

3. CLASSIC LIE GROUPS AND ALGEBRAS

During this lecture, we itemize the classic Lie groups and their algebras.

3.1. Group $SO(n)$ and algebra $o(n; \mathbb{R})$. Consider the transformations of n -dimensional Euclidean space $\mathbf{e} \rightarrow \mathbf{e}' = u \cdot \mathbf{e}$, see (17). The orthogonality condition (15) for the basis \mathbf{e}' in the matrix form reads $u^T u = \mathbf{1}$, where T is the matrix transposition.

Definition 11. *Set of $n \times n$ matrices with real entries satisfying*

$$(29) \quad u^T u = \mathbf{1}, \quad \det(u) = 1$$

is called the group $SO(n)$ (Special Orthogonal).

Let u_1 and u_2 satisfy (29). Then

$$(30) \quad (u_1 u_2)^T (u_1 u_2) = u_2^T u_1^T u_1 u_2 = \mathbf{1}, \quad \det(u_1 u_2) = \det(u_1) \det(u_2) = 1,$$

i.e. $u_1 u_2$ satisfy (29) as well. The group unity element is the matrix unity, group inverse element is the matrix inverse.

To find the corresponding Lie algebra, consider the neighborhood of the group unity (tangent space),

$$(31) \quad u = \mathbf{1} + x + O(x^2).$$

Conditions for x follows from (54):

$$(32) \quad x^T = -x.$$

Let x and y satisfy (32). Then

$$(33) \quad [x, y]^T = (xy - yx)^T = (xy)^T - (yx)^T = y^T x^T - x^T y^T = yx - xy = -[x, y].$$

We have verified that (32) is closed under the commutation.

Definition 12. *Algebra of anti-symmetric $n \times n$ matrices with real entries is called $o(n, \mathbb{R})$.*

See (22) as the example of $o(3, \mathbb{R})$. Anti-symmetric matrices with complex entries give $o(n, \mathbb{C})$.

A $n \times n$ matrix has n^2 elements, the number of independent equations in (29) is $\frac{n(n+1)}{2}$. Therefore, the dimension of $SO(n)$ as a variety equals to the dimension of $o(n)$ as the linear space equals to $\frac{n(n-1)}{2}$.

3.2. Similarity isomorphisms. The similarity transformation

$$(34) \quad x \mapsto x' \stackrel{\text{def}}{=} SxS^{-1}$$

is the isomorphism of a matrix algebra. Evidently,

$$(35) \quad S[xy]S^{-1} = SxyS^{-1} - SyxS^{-1} = SxS^{-1} SyS^{-1} - SyS^{-1} SxS^{-1} = [x'y'] ,$$

therefore $x \rightarrow SxS^{-1}$ is the homomorphism. Since the transformation is invertible, $x = S^{-1}x'S$, it is the isomorphism.

If S is an element of corresponding group, then $S\mathfrak{L}S^{-1} = \mathfrak{L}$ (this follows from (24)). In this case the isomorphism is called *automorphism*.

For instance, consider an element of the algebra $o(2\ell, \mathbb{C})$,

$$(36) \quad x = \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix} ,$$

where a, b, d are $\ell \times \ell$ matrices, in addition $a^T = -a, d^T = -d$. One may see,

$$(37) \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -i \mathbf{1} \\ \mathbf{1} & i \mathbf{1} \end{pmatrix} \Rightarrow x' = SxS^{-1} = \frac{1}{2} \begin{pmatrix} a + d + i(b + b^T) & a - d - i(b - b^T) \\ a - d + i(b - b^T) & a + d - i(b + b^T) \end{pmatrix}$$

Analyzing the structure of the right hand side, we see that $o(2\ell, \mathbb{C})$ is isomorphic to the algebra of block - matrices

$$(38) \quad x' = \begin{pmatrix} p & q \\ r & -p^T \end{pmatrix} , \quad q^T = -q, \quad r^T = -r .$$

Matrix S has the complex entries, therefore it is not the isomorphism of $o(2n, \mathbb{R})$. It is said that $o(2n, \mathbb{R})$ and the real regime of (38) are the different real forms on $o(2n, \mathbb{C})$.

Similarly, the algebra $o(2\ell + 1, \mathbb{C})$ is isomorphic to the algebra of block-matrices

$$(39) \quad \begin{pmatrix} 0 & n & m \\ -n^T & p & q \\ -m^T & r & -p^T \end{pmatrix} , \quad q^T = -q, \quad r^T = -r ,$$

where p, q, r are $\ell \times \ell$ matrices, m and n are $1 \times \ell$ row vectors.

3.3. **Group $SO(n, m)$.** Condition $u^T u = 1$ has an evident generalization:

$$(40) \quad u^T g u = g ,$$

for some matrix g . If g is generic³, this equation has the only solution $u = 1$. Matrices u form a parametric family if and only if the number of independent equations in (40) is less than n^2 . It may happen if

$$(41) \quad g^T = g \quad \text{or} \quad g^T = -g .$$

If $g^T = g$, one may always find the group similarity transformation

$$(42) \quad u \rightarrow S u S^{-1}$$

making g diagonal. Without loss of generality, diagonal g is given by

$$(43) \quad g = \text{diag}(\underbrace{1, 1, \dots, 1}_{n \text{ times}}, \underbrace{-1, -1, \dots, -1}_{m \text{ times}}) .$$

Definition 13. Group of $(n + m) \times (n + m)$ real matrices u satisfying

$$(44) \quad u^T g u = g , \quad \det(u) = 1 ,$$

where the signature of the metric g is given by (43), is called $SO(n, m)$.

The Minkovsky space has the signature $(+, -, -, -)$, its symmetry group is $SO(1, 3)$ – the Lorentz group.

Corresponding algebra is defined by relations

$$(45) \quad x^T g + g x = 0 .$$

As the complex algebra, it is isomorphic to $o(n + m, \mathbb{C})$. As the real algebra, it an additional real form on $o(n + m, \mathbb{C})$.⁴

³generic: relating to a whole group of similar things, rather than to any particular thing (Cambridge dictionary)

⁴Exercise: find the similarity transformation

$$(46) \quad \tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \mapsto \tau' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This similarity transformation relates the algebra of $SO(2)$ and the algebra of $SO(1, 1)$:

$$(47) \quad SO(2) : u(\theta) = e^{i\theta\tau} , \quad SO(1, 1) : u(\beta) = e^{\beta\tau'}$$

3.4. Group $SP(2n)$ and algebra $sp(2n)$. The alternative variant in (41) is $g^T = -g$. Such a matrix g is non-degenerative if its even-dimensional. Fix

$$(48) \quad g = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix},$$

where $\mathbb{1}_n$ is the $n \times n$ matrix unity.

Definition 14. *The group of $2n \times 2n$ matrices u satisfying*

$$(49) \quad u^T g u = g, \quad \det(u) = 1$$

where g is given by (48), is called $SP(2n)$ (Symplectic)

Considering the tangent space, we come to

Definition 15. *$sp(2\ell, \mathbb{C})$ is the Lie algebra of the complex block-matrices*

$$(50) \quad x = \begin{pmatrix} p & q \\ r & -p^T \end{pmatrix}, \quad q^T = q, \quad r^T = r.$$

4. CLASSIC LIE GROUPS AND ALGEBRAS (CONTD)

4.1. Group $SL(n)$ and algebra $sl(n, \mathbb{C})$.

Definition 16. Group of $n \times n$ complex matrices satisfying the single condition

$$(51) \quad \det(u) = 1$$

is called $SL(n)$ (Special Linear)

Definition 17. $sl(n, \mathbb{C})$ is the Lie algebra of traceless $n \times n$ matrices with complex coefficients.

Condition $\det(u) = 1$ provides the condition $\text{trace}(x) = 0$ since

$$(52) \quad \det e^x \equiv e^{\text{trace}(x)}$$

One may choose a particular form on $sl(n, \mathbb{C})$ such that

$$(53) \quad x^\dagger = -x,$$

where dagger stands for the Hermitian conjugation. This real form is the Lie algebra for the group $SU(n)$:

Definition 18. Set of $n \times n$ matrices u with complex valued entries satisfying

$$(54) \quad u^\dagger u = \mathbf{1}, \quad \det u = 1$$

is called the group $SU(n)$ (Special Unitary).

4.2. **Nicknames.** The complex algebras have the nicknames A, B, C, D :

$sl(\ell + 1, \mathbb{C})$	A_ℓ
$o(2\ell + 1, \mathbb{C})$	B_ℓ
$sp(2\ell, \mathbb{C})$	C_ℓ
$o(2\ell, \mathbb{C})$	D_ℓ

4.3. Adjoint representations. Recall, a representation of an algebra is a homomorphism π of algebra to endomorphisms of a linear space V . In addition, this linear space is called \mathfrak{L} -module. A Lie algebra \mathfrak{L} itself is a linear space. Define

$$(55) \quad \pi_0(x) y \stackrel{\text{def}}{=} [xy], \quad \forall y \in \mathfrak{L}$$

It is a homomorphism of algebra due to the Jacobi identity:

$$(56) \quad \pi_0([xy]) z = [[xy]z] = [x[yz]] - [y[xz]] = [\pi_0(x), \pi_0(y)] z .$$

Definition 19. *The adjoint representation of a Lie algebra is given by (55). The standard notation for $\pi_0(x)$ is $ad_{\mathfrak{L}}(x)$.*

Let a basis decomposition of a Lie algebra is chosen,

$$(57) \quad [x_{\alpha}, x_{\beta}] = \sum_{\gamma=1}^L c_{\alpha, \beta}^{\gamma} x_{\gamma}$$

where L is the dimension of \mathfrak{L} as the linear space. Then in this basis

$$(58) \quad ad(x_{\alpha}) x_{\beta} = \sum_{\gamma} c_{\alpha, \beta}^{\gamma} x_{\gamma} \Rightarrow \left(ad x_{\alpha} \right)_{\beta, \gamma} = c_{\alpha, \beta}^{\gamma}$$

Definition 20. $K(x, y) = \text{trace} \left(ad(x), ad(y) \right)$ is called the Killing form.

4.4. The algebra A_1 . Consider $A_1 = sl(2, \mathbb{C})$ in more details. The space of traceless matrices 2×2

$$(59) \quad x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

has the evident basis

$$(60) \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

The basis is valid both for $sl(2, \mathbb{R})$ and $sl(2, \mathbb{C})$. The algebraic relations in this basis are

$$(61) \quad [h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h .$$

Note, elements e and f both are eigenvectors of $ad(h)$.

Another useful basis of $sl(2, \mathbb{C})$ is

$$(62) \quad S_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One may see, $S_1 = \frac{e+f}{2}$, $S_2 = \frac{e-f}{2i}$, $S_3 = \frac{h}{2}$. The algebraic relations in S -basis are

$$(63) \quad [S_1, S_2] = iS_3, \quad [S_2, S_3] = iS_1, \quad [S_3, S_1] = iS_2.$$

Confer with (26). It follows, $\mathfrak{o}(3, \mathbb{C}) \cong sl(2, \mathbb{C})$ ($A_1 \cong B_1$).

Turn to adjoint representation of A_1 . Using (S_1, S_2, S_3) as basis vectors, the adjoint representation of A_1 (63) is

$$(64) \quad \text{ad}(S_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \text{ad}(S_2) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{ad}(S_3) = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Comparing with (22), we see $\text{ad}(S_j) = iT_j (\equiv L_j)$, see the definition of L right before relations (26).

4.5. Ideals. We will use notation $[\mathfrak{L}, \mathfrak{B}]$ for the subspace spanned by the set $\{[xy] \mid x \in \mathfrak{L}, y \in \mathfrak{B}\}$.

Definition 21. A subspace \mathfrak{I} of a Lie algebra \mathfrak{L} is an ideal if $[\mathfrak{L}, \mathfrak{I}] \subseteq \mathfrak{I}$.

An ideal is a subalgebra. Algebra $\mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$ is always an ideal (\mathfrak{L}' is called the derived algebra). Example: $[gl_\ell, gl_\ell] = sl_\ell$, sl_ℓ is the ideal of gl_ℓ .

The quotient space $\mathfrak{L}/\mathfrak{I}$ is the vector space of the equivalence classes in \mathfrak{L} : x and y belong to the same equivalence class if and only if $x - y \in \mathfrak{I}$. The quotient space is the Lie algebra since

$$(65) \quad [x + \mathfrak{I}, y + \mathfrak{I}] = [xy] + \mathfrak{I}$$

Example for physicists: consider

$$(66) \quad \mathfrak{L} \ni \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \right] = \begin{pmatrix} [aa'] & ab' + bd' - a'b - b'd \\ 0 & [dd'] \end{pmatrix}$$

Clearly, condition $d = 0$ fixes the ideal,

$$(67) \quad \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}, \quad \mathfrak{L}/\mathfrak{I} \ni \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ modulo } \mathfrak{I} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

Clearly, as a linear space $\mathfrak{L} = \mathfrak{I} \oplus (\mathfrak{L}/\mathfrak{I})$

Definition 22. \mathfrak{L} is Abelian if and only if $[\mathfrak{L}, \mathfrak{L}] = 0$.

Definition 23. A Lie algebra \mathfrak{L} is simple if it has no ideals and $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.

In particular, the Abelian algebras with $[\mathfrak{L}, \mathfrak{L}] = 0$ are not classified as simple.

Lemma 1. If \mathfrak{L} is a simple Lie algebra, then:

- (1) All representations are faithful, except when π is the trivial map $\pi : \mathfrak{L} \mapsto 0$.
- (2) $\pi(\mathfrak{L}) \subseteq sl(V)$.
- (3) The only one-dimensional representation is trivial.
- (4) If for any non-zero $x \in \mathfrak{L}$, $\pi(x) = 0$ then the representation is trivial.

Prove, for example, (4) and (3). For some $x \in \mathfrak{L}$ let $\mathfrak{L}^{(1)} = [\mathfrak{L}, x]$, $\mathfrak{L}^{(2)} = [\mathfrak{L}, \mathfrak{L}^{(1)}]$, etc., $\mathfrak{L}^{(n)} = [\mathfrak{L}, \mathfrak{L}^{(n-1)}]$. The algebra is simple and finite, therefore for some n we must meet $\mathfrak{L}^{(n)} = \mathfrak{L}$. If $\pi(x) = 0$, then $\pi(\mathfrak{L}^{(1)}) = 0$, etc., $\pi(\mathfrak{L}) = 0$. (4) is proven. Next, the one-dimensional representation is the representation by numbers, therefore for any x, y , $[xy] \neq 0$ we have $\pi([xy]) = 0 \Rightarrow \pi(\mathfrak{L}) = 0$. □

5. ROOT SYSTEMS OF CLASSICAL LIE ALGEBRAS

5.1. **The algebras** A_ℓ . Let $E_{ij} \in gl_{\ell+1}$ denote the matrix unit with single non-zero entry at row-column position (i, j) . Their key property is $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Let \mathfrak{H} be the set of all diagonal traceless matrices,

$$(68) \quad \mathfrak{H} \ni H = \{\text{diag}(x_1, \dots, x_{\ell+1}) \mid \sum_{i=1}^{\ell+1} x_i = 0\} = \sum_{i=1}^{\ell+1} x_i E_{ii}.$$

The algebra A_ℓ may be decomposed $A_\ell = \bigoplus_{i \neq j} \mathbb{C}E_{ij} \oplus \mathfrak{H}$. One may see, \mathfrak{H} is the Abelian subalgebra. Since $HE_{ij} = x_i E_{ij}$ and $E_{ij}H = E_{ij}x_j$,

$$(69) \quad (\text{ad}H)E_{ij} = (x_i - x_j)E_{ij} = \alpha_{ij}(H)E_{ij}, \quad \forall H \in \mathfrak{H}.$$

\mathfrak{H} is called a Cartan sub-algebra for A_ℓ . Note, in the adjoint representation the E_{ij} are simultaneously eigenvectors of all elements of \mathfrak{H} (\equiv the root vectors).

Equation (69) introduces an important object, $\alpha_{ij}(H) \stackrel{\text{def}}{=} x_i - x_j$. By definition, α_{ij} is the set of linear functions on \mathfrak{H} , or the dual space.

The dual space⁵ \mathfrak{H}^* is spanned by the linear functions ε_i defined by the action

$$(70) \quad \varepsilon_i : \text{diag}(x_1, \dots, x_{\ell+1}) \mapsto x_i.$$

Clearly, $\varepsilon_1 + \dots + \varepsilon_{\ell+1} = 0$, see (68), and we have

$$(71) \quad \alpha_{ij} = \varepsilon_i - \varepsilon_j.$$

The complete set of these linear forms corresponding to all non-diagonal elements E_{ij}

$$(72) \quad \Phi = \{\alpha_{ij} \mid i \neq j\}$$

is called *root system for* A_ℓ , $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ is called *root*. One may separate Φ to the sets of positive and negative roots,

$$(73) \quad \Phi_+ = \{\alpha_{ij} \mid i < j\}, \quad \Phi_- = \{\alpha_{ij} \mid i > j\}$$

⁵Comment for physicists. A dual space is a very common construction. In a primitive geometry, let a space V is spanned by its non-degenerative basis $\mathbf{e}_i[m]$. The dual basis is defined by the orthogonality conditions $(\mathbf{e}^i, \mathbf{e}_j) = \hbar \delta_{ij}$, where (\cdot, \cdot) is the scalar product. The dual space V^* is spanned by the vectors $\mathbf{e}^i[gm/s]$. Of course, V and V^* are isomorphic. We have used the units of length $[m]$ and momentum $[gm/s]$ in order to clarify that V and V^* are different spaces in general.

A suitable choice for the basis of \mathfrak{H}^* is the set of “simple roots” Δ :

$$(74) \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad (1 \leq i \leq \ell),$$

in terms of which the complete root system has the description

$$(75) \quad \Phi_{\pm} = \{\pm(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j) \mid 1 \leq i \leq j \leq \ell\}.$$

Note, the set Φ may be represented using the set Δ using only integer multipliers.

Due to the *one-to-one* correspondence between E_{ij} and λ_{ij} , we shall also introduce a more general (and more useful) notation for the root vectors

$$(76) \quad E_{ij} \stackrel{\text{def}}{=} E_{\alpha_{ij}}$$

i.e. we label the root vectors by the roots. The commutator of two root vectors with opposite roots is an element of Cartan subalgebra,

$$(77) \quad [E_{\alpha_{ij}}, E_{-\alpha_{ij}}] = E_{ii} - E_{jj} \stackrel{\text{def}}{=} H_{\alpha_{ij}} \in \mathfrak{H}.$$

In particular, H_{α_i} , $\alpha_i \in \Delta$, form a basis of \mathfrak{H} .

The dual space construction provides the notion of orthogonality. For any $\lambda \in \mathfrak{H}^*$ the elements $H \in \mathfrak{H}$: $\lambda(H) = 0$ form $\ell - 1$ dimensional subspace of \mathfrak{H} (a hyperplane orthogonal to λ).

Lemma 2. $\forall \lambda_0, \lambda_1 \neq m\lambda_0 \in \mathfrak{H}^* \exists H \in \mathfrak{H}$: $\lambda_0(H) \neq 0$ and $\lambda_1(H) = 0$.

Proof. Let \mathfrak{H}' be $\ell - 1$ -dimensional hyperplane orthogonal to λ_1 . Since \mathfrak{H}^* has ℓ -dimensional non-degenerative basis, this hyperplane is orthogonal to λ_0 if and only if $\lambda_0 \sim \lambda_1$. Otherwise, the hyperplane contains a family of elements $\lambda_0(H) \neq 0$. \square

Lemma 3. For any root $\alpha \in \Phi$ one may choose a set of elements of Cartan subalgebra H_1, H_2, \dots, H_m such that $\lambda(H_1)\lambda(H_2)\cdots\lambda(H_m) = 0$ for any root $\lambda \in \Phi$ except $\lambda = \pm\alpha$ ($\lambda = k\alpha$ in general).

Proof. Let $\lambda_1, \lambda_2, \dots$ be a set of distinct roots not parallel to α . Then H_1 comes from the previous lemma for α and λ_1 , H_2 comes from the previous lemma for α and λ_2 , etc. Since the number of elements of Φ is finite, the finite sequence of H_j exists. \square

Example: for the root system of A_2

$$(78) \quad \alpha_{12} = \epsilon_1 - \epsilon_2, \quad \alpha_{13} = \epsilon_1 - \epsilon_3, \quad \alpha_{23} = \epsilon_2 - \epsilon_3$$

Let us distinguish α_{12} . Then we choose H_1 orthogonal to α_{13} ,

$$(79) \quad H_1 = \text{diag}(1, -2, 1) : \quad \alpha_{13}(H_1) = 0 \text{ and } \alpha_{12}(H_1) = -3.$$

Next, choose H_2 orthogonal to α_{23} :

$$(80) \quad H_2 = \text{diag}(-2, 1, 1) : \quad \alpha_{23}(H_2) = 0 \text{ and } \alpha_{12}(H_2) = -3.$$

Evidently, $\alpha(H_1)\alpha(H_2) = 0$ for any $\alpha \in \{\pm\alpha_{13}, \pm\alpha_{23}\}$, and $\alpha(H_1)\alpha(H_2) \neq 0$ for $\alpha = \pm\alpha_{12}$.

Lemma 4. *Suppose that \mathfrak{J} is an ideal of A_ℓ . If for some root α we have $E_\alpha \in \mathfrak{J}$ or for some element $H \in \mathfrak{H}$ we have $H \in \mathfrak{J}$, then $\mathfrak{J} = A_\ell$.*

Proof. Let $H \in \mathfrak{J}$. Any root α , $\alpha(H) \neq 0$, provides $[E_\alpha, H] = -\alpha(H)E_\alpha \in \mathfrak{J}$. Therefore, $H \in \mathfrak{J}$ is equivalent to $E_\alpha \in \mathfrak{J}$ for some α .

Note next that

$$(81) \quad [E_\alpha, E_\beta] = \begin{cases} \pm E_{\alpha+\beta}, & \alpha + \beta \in \Phi, \\ H_\alpha & \alpha + \beta = 0, \\ 0, & \alpha + \beta \notin \Phi. \end{cases}$$

Let $\alpha \in \Phi_+$. Since the set of all positive roots can be obtained from any single positive root α by some sequence of addition (action by $\text{ad}E_{\alpha_i}$) and subtraction (action by $\text{ad}E_{-\alpha_i}$) of simple roots, all $E_{\Phi_+} \in \mathfrak{J}$. Applying $\text{ad}E_{-\Delta}$ to E_Δ , we obtain $\mathfrak{H} \in \mathfrak{J}$. Finally, $(\text{ad}E_{\Phi_-})\mathfrak{H} = E_{\Phi_-}$, therefore $\mathfrak{J} = A_\ell$. \square

Now the main theorem:

Theorem 1. *The Lie algebras A_ℓ , $\ell \geq 1$, are simple.*

Proof. Suppose,

$$(82) \quad A = H_0 + \sum_{\alpha \in \Phi} c_\alpha E_\alpha \in \mathfrak{J}$$

Let $c_\alpha \neq 0$ for some α . Then

$$(83) \quad A' = (\text{ad}H_1)(\text{ad}H_2) \cdots (\text{ad}H_m)A = u(c_\alpha E_\alpha + (-)^m c_{-\alpha} E_{-\alpha}) \in \mathfrak{J}, \quad u \neq 0.$$

where H_1, \dots, H_m come from lemma 3. In addition,

$$(84) \quad A'' = (\text{ad}H_\alpha)A' = 2u(c_\alpha E_\alpha - (-)^m c_{-\alpha} E_{-\alpha}) \in \mathfrak{J}$$

Therefore

$$(85) \quad A' + \frac{1}{2}A'' \sim E_\alpha \in \mathfrak{J}$$

and due to lemma 4 $\mathfrak{J} = A_\ell$. □

6. ROOT SYSTEMS OF CLASSICAL LIE ALGEBRAS (CONTD, HOME READING)

6.1. **Lie algebras** D_ℓ , ($\ell \geq 2$). We will use the definition (38). Let

$$(86) \quad x = \begin{pmatrix} p & q \\ r & -p^T \end{pmatrix} \leftrightarrow (p, q, r)$$

Write E_{ij} for the $\ell \times \ell$ matrix units (basis of p) and $E_{[ij]} = E_{ij} - E_{ji}$ (basis of q and p). Let \mathfrak{h} be the set of matrices of the form (notation (p, q, r))

$$(87) \quad H = (\text{diag}(x_1, \dots, x_\ell), 0, 0),$$

(this is obviously Abelian subalgebra) and define the linear functions on \mathfrak{h}

$$(88) \quad \varepsilon_i : (\text{diag}(x_1, \dots, x_\ell), 0, 0) \mapsto x_i$$

All ε_i are independent. Elementary computations give

$$(89) \quad \begin{aligned} [H, (E_{ij}, 0, 0)] &= (\varepsilon_i - \varepsilon_j)(H) (E_{ij}, 0, 0), \\ [H, (0, E_{[ij]}, 0)] &= (\varepsilon_i + \varepsilon_j)(H) (0, E_{[ij]}, 0), \\ [H, (0, 0, E_{[ij]})] &= -(\varepsilon_i + \varepsilon_j)(H) (0, 0, E_{[ij]}), \end{aligned}$$

This displays the root system for D_ℓ :

$$(90) \quad \Phi_\pm = \{\pm(\varepsilon_i - \varepsilon_j) \mid i < j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid i < j\}, \quad \Phi = \Phi_+ \cup \Phi_-.$$

For the simple roots Δ (basis of Φ_+) we may choose

$$(91) \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i < \ell, \quad \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell.$$

To check it, note that for Φ_+

$$(92) \quad \left. \begin{aligned} \varepsilon_i - \varepsilon_j &= (\alpha_i + \dots + \alpha_{j-1}), \\ \varepsilon_i + \varepsilon_j &= (\alpha_i + \dots + \alpha_{\ell-2}) + (\alpha_j + \dots + \alpha_\ell) \end{aligned} \right\} (i < j)$$

As well as in the case of A_ℓ , the set of all positive roots may be obtained from any single positive root α by some sequence of addition and subtraction of simple roots if $\ell \geq 3$. Then, repeating lemma 4, we come to

Theorem 2. *The Lie algebras D_ℓ , $\ell \geq 3$, are simple.*

D_1 is Abelian. In the case of D_2 the roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$ and there are no sequence of addition and subtraction translating α_1 into α_2 . Therefore, the lemma 4 does not work, and D_2 has two ideals: $D_2 = A_1 \oplus A_1$.

6.2. Lie algebras C_ℓ . We use the definition (50). As for D_ℓ ,

$$(93) \quad x = \begin{pmatrix} p & q \\ r & -p^T \end{pmatrix} \leftrightarrow (p, q, r)$$

Write E_{ij} for the $\ell \times \ell$ matrix units (basis of p) and $E_{\{ij\}} = E_{ij} + E_{ji}$ (basis of q and p). As in D_ℓ case,

$$(94) \quad H = (\text{diag}(x_1, \dots, x_\ell), 0, 0), \quad \varepsilon_i : H \mapsto x_i$$

All ε_i are independent. The commutation relations are

$$(95) \quad \begin{aligned} [H, (E_{ij}, 0, 0)] &= (\varepsilon_i - \varepsilon_j)(H) (E_{ij}, 0, 0), \\ [H, (0, E_{\{ij\}}, 0)] &= (\varepsilon_i + \varepsilon_j)(H) (0, E_{\{ij\}}, 0), \\ [H, (0, 0, E_{\{ij\}})] &= -(\varepsilon_i + \varepsilon_j)(H) (0, 0, E_{\{ij\}}), \end{aligned}$$

For the root system of C_ℓ these calculations show that

$$(96) \quad \Phi_\pm = \{\pm(\varepsilon_i - \varepsilon_j) \mid i < j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid i \leq j\}, \quad \Phi = \Phi_+ \cup \Phi_-.$$

The roots is the set Δ ,

$$(97) \quad \Delta : \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i < \ell, \quad \alpha_\ell = 2\varepsilon_\ell.$$

To check this assertion note that for Φ_+

$$(98) \quad \left. \begin{aligned} \varepsilon_i - \varepsilon_j &= (\alpha_i + \dots + \alpha_{j-1}), \\ \varepsilon_i + \varepsilon_j &= (\alpha_i + \dots + \alpha_{\ell-1}) + (\alpha_j + \dots + \alpha_\ell) \end{aligned} \right\} (i < j)$$

The structure of the root system provides

Theorem 3. *The Lie algebras C_ℓ , $\ell \geq 2$, are simple.*

Note, $C_1 = A_1$.

6.3. **Lie algebras B_ℓ .** We use the definition (39) for B_ℓ .

$$(99) \quad x = \begin{pmatrix} 0 & n & m \\ -n^T & p & q \\ -m^T & r & -p^T \end{pmatrix} \leftrightarrow (m, n; p, q, r).$$

Cartan subalgebra is an elements of p ,

$$(100) \quad H = (0, 0; \text{diag}(x_1, \dots, x_\ell), 0, 0), \quad \varepsilon_i : H \mapsto x_i.$$

Elementary computations give

$$(101) \quad \begin{aligned} [H, (0, 0; E_{ij}, 0, 0)] &= (\varepsilon_i - \varepsilon_j)(H)(0, 0; E_{ij}, 0, 0), \\ [H, (0, 0; 0, E_{[ij]}, 0)] &= (\varepsilon_i + \varepsilon_j)(H)(0, 0; 0, E_{[ij]}, 0), \\ [H, (0, 0; 0, 0, E_{[ij]})] &= -(\varepsilon_i + \varepsilon_j)(H)(0, 0; 0, 0, E_{[ij]}), \\ [H, (0, E_i; 0, 0, 0)] &= \varepsilon_i(H)(0, E_i; 0, 0, 0), \\ [H, (E_i, 0; 0, 0, 0)] &= -\varepsilon_i(H)(E_i, 0; 0, 0, 0), \end{aligned}$$

The root system for B_ℓ is evident,

$$(102) \quad \Phi_\pm = \{\pm(\varepsilon_i - \varepsilon_j) \mid i < j\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid i < j\} \cup \{\varepsilon_i\}, \quad \Phi = \Phi_+ \cup \Phi_-,$$

the basis of Φ_+ is

$$(103) \quad \Delta : \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i < \ell, \quad \alpha_\ell = \varepsilon_\ell.$$

To check this assertion note that for $i < j$

$$(104) \quad \begin{aligned} \varepsilon_i - \varepsilon_j &= (\alpha_i + \dots + \alpha_{j-1}), \\ \varepsilon_i + \varepsilon_j &= (\alpha_i + \dots + \alpha_\ell) + (\alpha_j + \dots + \alpha_\ell), \\ \varepsilon_i &= (\alpha_i + \dots + \alpha_\ell). \end{aligned}$$

Again,

Theorem 4. *The Lie algebras B_ℓ , $\ell \geq 2$, are simple.*

Recall, $B_1 = A_1$.

A TUTORIAL

1). Parameterize an element of $SO(1, 1)$.

Solution: the group relation gives

$$(105) \quad u = \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix}, \quad u_{11}^2 - u_{12}^2 = 1.$$

One parameterization: $u_{11} = \cosh \theta$ and $u_{12} = \sinh \theta$. Advantage: $u(\theta)u(\theta') = u(\theta + \theta')$.

Another parameterization: $u_{11} = \frac{1}{\sqrt{1 - \beta^2}}$ and $u_{12} = \frac{\beta}{\sqrt{1 - \beta^2}}$, $\beta = \frac{v}{c}$. The transformation relates an initial frame of reference and another one, moving with the speed β .

2). Prove the similarity of $u^T G u = G$ with arbitrary non-degenerate G and $v^T g v = g$ with diagonal g .

Let N be dimension of G . Let v_j be eigenvectors of G :

$$(106) \quad G v_j = \alpha_j v_j, \quad j = 1, \dots, N,$$

where α_j are real, but not necessary positive. Let them be normalized

$$(107) \quad v_j^T v_k = \frac{\delta_{jk}}{|\alpha_k|}.$$

Let $\epsilon_j = \text{sign}(\alpha_j)$. Combine the column-vectors v_j into the matrix:

$$(108) \quad B = (v_1, v_2, \dots, v_N)$$

We have

$$(109) \quad GB = (\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_N v_N)$$

Next,

$$(110) \quad B^T = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_N^T \end{pmatrix}, \quad B^T G B = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_N) \equiv g.$$

Inverse relation is $G = (B^T)^{-1} g B^{-1}$. Then $u^T G u = G$ may be rewritten

$$(111) \quad (B^{-1} u B)^T g (B^{-1} u B) = g.$$

B gives the similarity transformation.

3). Find the decomposition of the Lorentz group's algebra into two ideals.

The algebra is defined by

$$(112) \quad x^T g + g x = 0, \quad g = \text{diag}(1, -1, -1, -1).$$

Basis of matrix units E_{ij} , $ij = 0, 1, 2, 3$ is useful here. The basis of the algebra is given by six generators. Three of them are symmetric,

$$(113) \quad K_1 = E_{01} + E_{10}, \quad K_2 = E_{02} + E_{20}, \quad K_3 = E_{03} + E_{30},$$

three of them are antisymmetric:

$$(114) \quad T_1 = E_{32} - E_{23}, \quad T_2 = E_{13} - E_{31}, \quad T_3 = E_{21} - E_{12},$$

The algebra is

$$(115) \quad [K_1, K_2] = -T_3, \quad [K_1, T_2] = K_3, \quad [T_1, T_2] = T_3$$

and cyclic permutation. Let then

$$(116) \quad S_j^\pm = \frac{1}{2}(K_j \pm iT_j)$$

Then

$$(117) \quad [S_1^\pm, S_2^\pm] = iS_3^\pm \quad \text{and cyclic perms},$$

and $[S_j^\pm, S_k^\mp] = 0$.

7. A FEW FACTS FROM THE LINEAR ALGEBRA

Let A be n by n matrix. It acts in n -dimensional vector space V . We will call A the endomorphism of V .

More exactly, endomorphism A is a linear transformation of V . A has a matrix form with respect to a basis v_j of V . In the basis,

$$(118) \quad A v_j = \sum_k v_k A_{kj} .$$

If one chooses another basis, the same endomorphism will have other matrix elements.

Characteristic polynomial of endomorphism A is

$$(119) \quad P(\lambda) = \det(\lambda \cdot \mathbf{1} - A) = \prod_{j=1}^m (\lambda - \lambda_j)^{n_j} , \quad \sum_j n_j = n .$$

Since $P(A) \equiv 0$,

$$(120) \quad \prod_{j=1}^m (A - \lambda_j)^{n_j} \equiv 0 .$$

It is known from the Linear Algebra, any endomorphism A produces the spectral decomposition of V :

$$(121) \quad V = V_1 \oplus V_2 \oplus \dots \oplus V_m ,$$

where V_j are defined by

$$(122) \quad (A - \lambda_j)^{n_j} V_j = 0 , \quad \dim V_j = n_j .$$

In particular, if $n_j = 1$, V_j is one-dimensional, $V_j = cv_j$, v_j is called eigenvector.

Let $n_j \neq 0$ and let v_k be n_j -dimensional basis of V_j . Then

$$(123) \quad (A - \lambda_j)v_k = \sum_{l=1}^{n_j} \mu_{j;kl} v_l .$$

From (122) it follows, $\mu_j^{n_j} \equiv 0$. Such matrices are called nilpotent matrices (or Jordanian cells). Example,

$$(124) \quad \mu = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} , \quad \mu^2 \equiv 0 .$$

On the subspace V_j , A is the sum of unity λ_j and the nilpotent matrix μ . Gathering all the subspaces we get

$$(125) \quad \begin{array}{rcc} AV_j & = & \lambda_j V_j + \mu_j V_j \\ \downarrow & & \downarrow \quad \downarrow \\ AV & = & A_s V + A_n V \end{array}$$

where A_s may be diagonalised (all elements of V_j are eigenvectors of A_s), A_n is nilpotent.

Definition 24. *An endomorphism is semi-simple if its eigenvectors form a basis for V .*

Definition 25. *An endomorphism A is nilpotent if for some m , $A^m = 0$.*

By construction (125), any endomorphism has a unique decomposition $A = A_s + A_n$ (Jordan-Chevalley decomposition), where A_s is semi-simple, A_n is nilpotent and $[A_s, A_n] = 0$.

Lemma 5. *Let $[A, B] = 0$ and A has the spectral decomposition $\bigoplus_{j=1}^m V_j$. Then $BV_j \subseteq V_j$*

Proof: let v_k , $k = 1, \dots, \dim V_j$, be a basis of V_j . Then

$$(126) \quad (AB - BA)v_k = (A - \lambda_j)Bv_k - \sum_l \mu_{kl} Bv_l = 0 \Rightarrow (A - \lambda_j)^{n_j} Bv_k = 0 \Rightarrow BV_j \subseteq V_j.$$

□

It follows,

Lemma 6. *Commuting semi-simple endomorphisms may be simultaneously diagonalised.*

Definition 26. *A set \mathfrak{S} of endomorphisms of a vector space V is reducible with respect to subspace $W \subset V$ if $AW \subseteq W$ for any $A \in \mathfrak{S}$.*

Lemma 7. (Schur) *Suppose a set \mathfrak{S} of endomorphisms of V is irreducible. Then any endomorphism B : $[A, B] = 0 \forall A \in \mathfrak{S}$, is a multiple of unity.*

Proof. Suppose the spectral decomposition of B is $V_1 \oplus V_2 \oplus \dots$. Then all $A \in \mathfrak{S}$ must have V_j as invariant subspaces, therefore the spectral decomposition of B is $V \Rightarrow B = c \cdot 1 + B_n$, B_n -nilpotent. Let further $V' = B_n V \Rightarrow \forall A \in \mathfrak{S}$, $AV' = AB_n V = B_n AV \subseteq V' \Rightarrow \mathfrak{S}$ is reducible unless $B_n = 0$. □

Turn back to the Lie algebras.

Theorem 5. (Jordan-Chevalley) *Let \mathfrak{L} be simple finite-dimensional Lie algebra. Any $x \in \mathfrak{L}$ has unique decomposition $x = s + n$, its semi-simple and nilpotent elements $s, n \in \mathfrak{L}$, such that for any representation π*

$$(127) \quad \pi(x) = \pi(x)_s + \pi(x)_n \quad : \quad \pi(x)_s = \pi(s), \quad \pi(x)_n = \pi(n).$$

We do not need a complete proof of the theorem, we need actually its consequence: if $\text{ad}_{\mathfrak{L}}x$ is semi-simple/nilpotent, then $\pi(x)$ is semi-simple/nilpotent.

Lemma 8. *Let A be an endomorphism of the vector space V . If A is semi-simple/nilpotent so is $\text{ad}A$.*

Proof. (i) If A is semi-simple, choose the basis of its eigenvectors v_i as the basis of V . Fix the basis of $\text{End}(V)$: $E_{ij}v_i = v_j$. Then $(\text{ad}A)E_{ij} = (a_i - a_j)E_{ij}$, that is, $\text{ad}A$ is diagonal.

(ii) $(\text{ad}A)^n B$ is a sum of the terms $A^k B A^{n-k}$, $0 \leq k \leq n$. It follows immediately that $\text{ad}A$ is nilpotent if A is. □

End of lecture -- Next lecture

Lemma 9. *Let $A \in \text{End}(V)$ with $A = A_s + A_n$ its Jordan-Chevalley decomposition. Then the decomposition of $\text{ad}A$ is $\text{ad}A = \text{ad}A_s + \text{ad}A_n$.*

Proof. The previous lemma shows that $\text{ad}A_s$ and $\text{ad}A_n$ are semi-simple and nilpotent respectively. One may verify, they commute – therefore the uniqueness condition for the decomposition is satisfied. □

Now we are ready to prove Jordan-Chevalley theorem in its weak form. Let $\text{ad}_{\mathfrak{L}}x$ is simple. Then $\text{ad}_{\pi(\mathfrak{L})}\pi(x)$ is as well simple. Therefore $\pi(x)$ must be simple, otherwise $\text{ad}_{\pi(\mathfrak{L})}\pi(x)$ would have a nilpotent part. □

Definition 27. *A set \mathfrak{S} of endomorphisms of a vector space V is completely reducible if V has the direct decomposition $V = V_1 \oplus V_2 \oplus \dots \oplus V_m$ such that $AV_j \subseteq V_j$ for any $A \in \mathfrak{S}$ and \mathfrak{S} is irreducible over each V_j*

Theorem 6. (First theorem of H. Weyl) *Let \mathfrak{L} be a finite-dimensional simple Lie algebra and let $\pi: \mathfrak{L} \mapsto \text{sl}(V)$ be a finite dimensional representation of \mathfrak{L} . Then $\pi(\mathfrak{L})$ is completely reducible.*

Remark on Proof: what is the problem. Let $W \subset V$ be an invariant subspace and $W' = V/W$. Then in general

$$(128) \quad \pi(x)W = A_x W, \quad \pi(x)W' = B_x W' + \lambda_x W,$$

so that $\pi(x)$ has the block-triangular structure

$$(129) \quad \pi(x) = \begin{pmatrix} B_x & \lambda_x \\ 0 & A_x \end{pmatrix}, \quad V = \begin{pmatrix} W' \\ W \end{pmatrix}.$$

Dimensions of W and W' are in general different. The point is that $\lambda_x \neq 0$. But π is the representation of \mathfrak{L} ,

$$(130) \quad \pi([xy]) = \begin{pmatrix} [B_x, B_y] & \lambda_{[xy]} \\ 0 & [A_x, A_y] \end{pmatrix},$$

where

$$(131) \quad \lambda_{[xy]} = B_x \lambda_y - B_y \lambda_x + \lambda_x A_y - \lambda_y A_x$$

Everything is more or less clear with the A_x and B_x blocks. But λ_x -block is suspicious, and if $\dim W \neq \dim W'$, λ_x are non-square matrices. Suppose, a basis of algebra is chosen, then the set of equations (131) is a set of *homogeneous linear* equations for all λ_x , $x \in \mathfrak{L}$. As it is known, solutions of any set of linear homogeneous equations have a basis, so that an arbitrary solution is a linear combination of the basic ones.

Weyl's analytical proof of his theorem is just a general solution of the linear system (131). It is:

$$(132) \quad \lambda_x = B_x Q - Q A_x$$

where Q is an arbitrary constant $\dim W' \times \dim W$ matrix. Basis solutions correspond to matrix-unit elements of Q . Then

$$(133) \quad \pi(x) = \begin{pmatrix} B_x & B_x Q - Q A_x \\ 0 & A_x \end{pmatrix} = \begin{pmatrix} 1 & -Q \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} B_x & 0 \\ 0 & A_x \end{pmatrix} \cdot \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix}.$$

The outmost matrices in the right hand side here are the inverse, they define the change of the basis

$$(134) \quad \begin{pmatrix} W'' \\ W \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} W' \\ W \end{pmatrix} = \begin{pmatrix} W' - QW \\ W \end{pmatrix}.$$

In the new basis

$$(135) \quad \pi(x) = \begin{pmatrix} B_x & 0 \\ 0 & A_x \end{pmatrix}$$

The existence of a basis such that $\pi(x)$ is block-diagonal, is the subject of the Weyl theorem.

8. REPRESENTATION THEORY OF A_1

8.1. **Irreducible A_1 modules.** Recall, the A_1 algebra is defined by (61):

$$(136) \quad [he] = 2e, \quad [hf] = -2f, \quad [ef] = h.$$

The crucial point is that h is semi-simple (in the fundamental and in the adjoint representations), therefore for any representation $\pi(h)$ is semi-simple. Then we may choose the basis $\{v_0, v_1, \dots, v_n\}$ of a $n+1$ -dimensional module V such that $\pi(h)v_j = \lambda_j v_j$. We denote the set of all eigenvalues λ_j by $P(\pi)$ to exhibit their dependence on the representation. This gives the direct sum decomposition of V :

$$(137) \quad V = \bigoplus_{\lambda \in P(\pi)} V_\lambda$$

where

$$(138) \quad V_\lambda = \{v \in V \mid \pi(h)v = \lambda v\}$$

For the brevity, we will omit π using h instead of $\pi(h)$.

Definition 28. *The eigenvalues λ of h , acting in the module V , are called the weights of the representation. The corresponding subspaces V_λ are called weight spaces and their elements weight vectors.*

Lemma 10. *If $v \in V_\lambda$, then either $ev = 0$ or $ev \in V_{\lambda+2} \neq 0$. Similarly, either $fv = 0$, or $fv \in V_{\lambda-2} \neq 0$.*

Proof. In the representation, $[]$ is the commutator, $[he] = 2e$ is equivalent to $he = e(h+2)$

$$(139) \quad he v = h(ev) = e(h+2)v = e(\lambda+2)v = (\lambda+2)(ev)$$

and similarly for fv . □

Since the module is finite-dimensional, there must exist a weight λ , and a non-zero weight space V_λ , for which $ev = 0$, for all $v \in V_\lambda$. Choose $v_0 \in V_\lambda$; such a vector is called a *highest weight vector*. Define the sequence v_0, v_1, \dots, v_J by $v_{m+1} = fv_m$. It must terminate for some integer J .

Lemma 11. *The action of A_1 on the set of weight vectors $\{v_0, v_1, \dots, v_J\}$ is*

$$(140) \quad \begin{aligned} f v_m &= v_{m+1}, \\ e v_m &= m(J+1-m)v_{m-1}, \\ h v_m &= (J-2m)v_m \quad \Rightarrow \quad [v_m \in V_{J-2m}] \end{aligned}$$

and these vectors form a basis for an irreducible module.

Proof is just arithmetics. We have highest weight vector v_0 : $ev_0 = 0$ and $hv_0 = \lambda v_0$ for some λ .

Then $v_m \stackrel{\text{def}}{=} f^m v_0$. Relation $[hf] = -2f$ may be rewritten $hf = f(h-2)$ (it was used in the previous lemma), moreover $hf^m = f^m(h-2m)$, therefore $hv_m = (\lambda - 2m)v_m$. The only point is to guess e : $ev_m = c_m v_{m-1}$. To calculate c_m , consider

$$(141) \quad e v_{m+1} = c_{m+1} v_m = e f v_m = (h + fe)v_m = (\lambda - 2m)v_m + c_{m-1} v_m$$

therefore $c_{m+1} = (\lambda - 2m) + c_{m-1}$. This is the simple recursion relation, its solution is $c_m = m(\lambda + 1 - m) + c_0$. Since $eu_0 = 0$, $c_0 \equiv 0$. We now have

$$(142) \quad v_m = f^m u_0, \quad h v_m = (\lambda - 2m)v_m, \quad e v_m = m(\lambda + 1 - m)v_{m-1}$$

Let now v_J is “the last” vector in the sequence v_0, \dots, v_J : $f v_J = 0$. It means,

$$(143) \quad 0 = e f v_J = (fe + h)v_J = (c_J + (\lambda - 2J))v_J = (J + 1)(\lambda - J)v_J$$

therefore $\lambda = J$.

From the previous lemma we see that the weight vectors $\{v_0, \dots, v_J\}$ are linearly independent, hence they span the whole irreducible module, since they would otherwise span a proper submodule. □

End of lecture -- Next lecture

For a given basis $\{v_0, \dots, v_J\}$, matrix forms of e, f, h may be restored with the help of (118).

So we have proved the following classification theorem for all finite-dimensional irreducible A_1 modules:

Theorem 7. *For each positive integer J , there is a unique (up to isomorphism) $(J + 1)$ -dimensional irreducible A_1 module V_J . Its weight spaces are all one-dimensional, with weights*

$J, J-2, \dots, -J$. The entire module is generated from a highest weight vectors v_0 by the action of the algebra.

An exercise (example of how it works): Let $J = 1$ ($J = 0$ gives trivial one-dimensional representation, the smallest non-trivial is $J = 1$), and choose

$$(144) \quad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then relations (140) correspond exactly to (60).

8.2. Alternative parameterization. Sometimes people use another form of (140), choosing the weight as the index of the basis vectors. Then

$$(145) \quad V_J = \{v_J, v_{J-2}, \dots, v_{-J}\}$$

and

$$(146) \quad hv_\lambda = \lambda v_\lambda, \quad ev_\lambda = d_\lambda v_{\lambda+2}, \quad fv_\lambda = c_\lambda v_{\lambda-2},$$

where

$$(147) \quad d_\lambda c_{\lambda+2} = \frac{1}{4}(J-\lambda)(J+2+\lambda).$$

The definition (140) corresponds to $c_\lambda = 1$, an alternative symmetric parameterization is

$$(148) \quad d_\lambda = c_{\lambda+2} = \frac{1}{2}\sqrt{(J-\lambda)(J+2+\lambda)}.$$

In this normalization, $f = e^T$.

8.3. Space of homogeneous polynomials. A very simple construction reproduces all the finite dimensional representations of A_1 . Let

$$(149) \quad e = x_1 \frac{\partial}{\partial x_2}, \quad f = x_2 \frac{\partial}{\partial x_1}, \quad h = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2},$$

where e, f, h are operators acting in the space of functions of x_1, x_2 . At the first, one may verify

$$(150) \quad [ef] = h, \quad [he] = 2e, \quad [hf] = -2f,$$

i.e. they are some “generalized” representation of A_1 (generalized – because the space of functions is not a finite-dimensional one).

Now,

$$(151) \quad v_0 = x_1^J$$

is definitely a highest weight vector, its weight is J . Applying f to v_0 , we obtain (normalization differs slightly from (140))

$$(152) \quad v_m = x_1^{J-m} x_2^m, \quad v_J = x_2^J.$$

Therefore, the submodule with the highest weight J is formed by the polynomials

$$(153) \quad P_J(x_1, x_2) = \sum_{m=0}^J c_m x_1^{J-m} x_2^m$$

– they are called homogeneous polynomials of the power J . Thus the space of entire functions of two variables is isomorphic to the infinite direct sum $\bigoplus_{J=0}^{\infty} V_J$.

End of lecture -- Next lecture

8.4. Decomposition of A_1 modules. For any finite-dimensional A_1 module V , there must exist at least one highest weight vector u_0 . If the module V' which it generates does not exhaust V , then we may use the Weyl theorem to get the direct sum decomposition $V = V' \oplus V''$. V'' is an A_1 module of smaller dimension; proceeding inductively we obtain a decomposition of V into irreducible highest weight A_1 modules. So we have classified all finite-dimensional A_1 modules: they are finite direct sums of the irreducible modules V_J described in the previous theorem:

$$(154) \quad V = V_{J_1} \oplus V_{J_2} \oplus \dots \oplus V_{J_m}$$

This statement may be verified on an important physics inspired construction.

Let \mathfrak{L}_1 and \mathfrak{L}_2 be two identical copies of the same Lie algebra \mathfrak{L} . For any $x \in \mathfrak{L}$ let $(x)_1$ stands for the element x in \mathfrak{L}_1 and $(x)_2$ stands for the same element x in \mathfrak{L}_2 . The copies \mathfrak{L}_1 and \mathfrak{L}_2 are independent, $[\mathfrak{L}_1, \mathfrak{L}_2] \equiv 0$. Define

$$(155) \quad \Delta(x) = (x)_1 + (x)_2.$$

Since $[\mathfrak{L}_1, \mathfrak{L}_2] = 0$, it is easy to see

$$(156) \quad [\Delta(x), \Delta(y)] = ([xy])_1 + ([xy])_2,$$

i.e Δ is a specific algebra homomorphism. As a map, $\Delta : \mathfrak{L} \mapsto \mathfrak{L} \oplus \mathfrak{L}$.

Let further π_1 be a representation of \mathfrak{L}_1 in $\text{End}(V^{(1)})$ and π_2 be a representation of \mathfrak{L}_2 in $\text{End}(V^{(2)})$. The representation space for $\Delta(\mathfrak{L})$ is then the *tensor product* $V^{(1)} \otimes V^{(2)}$: the linear space spanned by the vectors $v_i \otimes w_j$, $v_i \in V^{(1)}$ and $w_j \in V^{(2)}$. By definition, $\Delta(x)$ acts in $V^{(1)} \otimes V^{(2)}$ as⁶

$$(157) \quad \Delta(x) \cdot v_i \otimes w_j \stackrel{\text{def}}{=} \pi_1(x)v_i \otimes w_j + v_i \otimes \pi_2(x)w_j .$$

Since Δ is the algebra homomorphism, this is a representation of \mathfrak{L} .

Example: let $V^{(1)}$ and $V^{(2)}$ be the A_1 -modules with the highest weights 1. The basis of $V_1 \otimes V_1$ is therefore

$$(158) \quad v_0 \otimes v_0 , \quad v_0 \otimes v_1 , \quad v_1 \otimes v_0 , \quad v_1 \otimes v_1 ,$$

where v_0, v_1 are defined by (140) with $J = 1$. The highest weigh vector here is $v_0 \otimes v_0$:

$$(159) \quad \Delta(e)v_0 \otimes v_0 \stackrel{\text{def}}{=} ev_0 \otimes v_0 + v_0 \otimes ev_0 = 0 .$$

Apply now $\Delta(f)$ to $v_0 \otimes v_0$ to reconstruct its irreducible sub-module:

$$(160) \quad \begin{aligned} \Delta(f)v_0 \otimes v_0 &= v_0 \otimes v_1 + v_1 \otimes v_0 , \\ \Delta(f)^2 v_0 \otimes v_0 &= 2v_1 \otimes v_1 , \\ \Delta(f)^3 v_0 \otimes v_0 &= 0 . \end{aligned}$$

This sub-module corresponds to $J = 2$ (exercise: check the values of $\Delta(h)$),

$$(161) \quad V_2 : \quad u_0 = v_0 \otimes v_0 , \quad u_1 = v_0 \otimes v_1 + v_1 \otimes v_0 , \quad u_2 = 2v_1 \otimes v_1 .$$

This basis has dimension 3 while $V_1 \otimes V_1$ has dimension 4. The lost vector must have form $w_0 = av_0 \otimes v_1 + bv_1 \otimes v_0$. It must be a highest weight vector, and the equation $\Delta(e)w_0 = 0$ gives $a + b = 0$, therefore

$$(162) \quad V_0 : \quad w_0 = v_0 \otimes v_1 - v_1 \otimes v_0 .$$

So, we have the decomposition of the reducible space $V_1 \otimes V_1$ into irreducible V_2 with the basis u_0, u_1, u_2 and irreducible (trivial) space V_0 with the single basis vector w_0 . Multiplication table for the dimensions is

$$(163) \quad 2 \times 2 = 3 + 1 .$$

⁶More precisely, it is definition of $(\pi_1 \otimes \pi_2)\Delta(x)$. Sometimes people use the notation $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Comment for physicists: Δ comes from quantum mechanics. Consider two identical particles, the first one has coordinate x_1 and momentum p_1 , the second one x_2 and p_2 . In quantum mechanics x_j, p_j form the *algebra of observables*. Note, the pair x_1, p_1 commutes with the pair x_2, p_2 (they are independent). The whole system has the coordinate of the mass center $X = (x_1 + x_2)/2$ and the total momentum

$$(164) \quad P = p_1 + p_2 .$$

The wave function of the system is a function of two variables, $\Psi(x_1, x_2)$. The total momentum acts as follows:

$$(165) \quad P\Psi(x_1, x_2) = i\hbar \frac{\partial}{\partial x_1} \Psi(x_1, x_2) + i\hbar \frac{\partial}{\partial x_2} \Psi(x_1, x_2)$$

Equation (155) is the formalization of (164), equation (157) is the formalization of (165), the tensor product $V^{(1)} \otimes V^{(2)}$ is the formalization of $\Psi(x_1, x_2)$.

TUTORIAL II

Question was: direct sums, reducibility/irreducibility/complete reducibility.

Direct sum of linear spaces. Let v_1, v_2, \dots, v_n be a basis of n -dimensional vector space V . Vectors $u = \sum_{i=1}^m c_i v_i$ form the m -dimensional subspace $V_1 \subset V$, $u \in V_1$. The complementary

basis vectors $w = \sum_{i=m+1}^n c_i v_i$ form the $(n - m)$ -dimensional subspace $V_2 \subset V$, $w \in V_2$.

Any vector $v \in V$ has the *unique* decomposition with respect to V_1 and V_2 ,

$$v = \sum_{i=1}^n c_i v_i = \sum_{i=1}^m c_i v_i + \sum_{i=m+1}^n c_i v_i = u + w, \quad u \in V_1, \quad w \in V_2.$$

This *uniqueness* is the meaning of the *direct sum* decomposition

$$V = V_1 \oplus V_2.$$

Note, the basis is not only something like $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, but in general

$$v_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \quad v_2 = \begin{pmatrix} b \\ d \end{pmatrix}, \quad ad - bc \neq 0.$$

Of course, any basis defines the maximal direct sum decomposition

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_n,$$

into *one-dimensional* subspaces $V_i = c_i v_i$. In some sense it is the definition of basis.

Reducibility-irreducibility-complete reducibility of set of endomorphisms. A few comments.

1). Let \mathfrak{S} consists on a single matrix A . A is of course reducible with respect to any its eigenvector v_i since $Av_i = a_i v_i \in V_i$, $V_i = c v_i$.

2). Therefore, to get the irreducibility, we must take more than one matrix in the set.

3). What is the complete reducibility. Repeat the definition. A set \mathfrak{S} is completely reducible if there is a special basis in V such that any $A \in \mathfrak{S}$ has the block-diagonal structure

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & \ddots \end{pmatrix}$$

where the blocks A_1, A_2, \dots , are some matrices. This structure must be the same for all $A \in \mathfrak{S}$. Block A_j acts in the space $V_j \in V$, and $V = V_1 \oplus V_2 \oplus \dots$.

The point is: the set is completely reducible if it *defines* the direct sum decomposition of the vector space V .

Example (in the block-diagonal form). $\mathfrak{S} = \{A, B\}$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Choose the basis

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

We see, $Av_1 = 0$ and $Bv_1 = v_1$: \mathfrak{S} is reducible with respect to v_1 . Now let $u = av_2 + bv_3 \in V_{23} \subset V$ (a subspace spanned by v_2 and v_3).

$$Au = av_2 - bv_3 \in V_{23}, \quad Bu = bv_2 \in V_{23}$$

therefore \mathfrak{S} is reducible with respect to V_{23} . Thus, the decomposition $V = V_1 \oplus V_{23}$ reduces \mathfrak{S} . V_1 is one-dimensional, it is irreducible by definition. Turn to the block V_{23} where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question: does there exist a subspace $V' \subset V_{23}$ such that $AV' \subseteq V'$ and $BV' \subseteq V'$? If so, \mathfrak{S} is reducible with respect to V' .

Since $\dim V_{23} = 2$, $\dim V'$ maybe only 1 (otherwise, $V' = V_{23}$ or $V' = \emptyset$). But A and B have no common eigenvectors, therefore one-dimensional V' does not exist. Block V_{23} is irreducible.

Therefore, the initial set \mathfrak{S} is completely reducible (it is given in the reduced form).

4). Consider another example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

These two matrices have a single common eigenvector $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, the set is reducible with respect to the [one-dimensional] space $V_1 = cv_1$.

We may write the decomposition of V , $V = V_1 \oplus V_2$, where $V_2 = c \begin{pmatrix} 1 \\ \beta \end{pmatrix}$. You may verify, the set of A, B has no common eigenvector of the type V_1 . It is the example of:

- \mathfrak{S} is reducible with respect to V_1 and therefore \mathfrak{S} is not irreducible.
- \mathfrak{S} is not completely reducible because of there are no V_2 such that $\mathfrak{S}V_2 \subseteq V_2$ and $V = V_1 \oplus V_2$.

9. NILPOTENT AND SOLVABLE ALGEBRAS

Let us recollect the purposes of first part of our course:

- 1 Structure theory: To classify the finite-dimensional simple Lie algebras and
- 2 Representation theory: To classify their representations.

To achieve our both purposes, we need a more information about subalgebras of a simple Lie algebra.

9.1. **Definitions.** Given a Lie algebra \mathfrak{L} , define a descending chain of ideals by

$$(166) \quad \mathfrak{L}^1 = [\mathfrak{L}, \mathfrak{L}], \quad \mathfrak{L}^2 = [\mathfrak{L}, \mathfrak{L}^1], \quad \dots \quad \mathfrak{L}^n = [\mathfrak{L}, \mathfrak{L}^{n-1}].$$

The power sign stands for “adjoint power”.

Definition 29. *The chain of ideals (166) is the lower central series.*

Definition 30. *A Lie algebra \mathfrak{L} is nilpotent if $\mathfrak{L}^m = 0$ for some integer m .*

Do not mix the notion of nilpotent endomorphism and nilpotent algebra!

Given a Lie algebra \mathfrak{L} , let

$$(167) \quad \mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}], \quad \mathfrak{L}'' = [\mathfrak{L}', \mathfrak{L}'], \quad \dots \quad \mathfrak{L}^{(n)} = [\mathfrak{L}^{(n-1)}, \mathfrak{L}^{(n-1)}].$$

Definition 31. *The chain of ideals (167) is the derived series.*

Definition 32. *A Lie algebra \mathfrak{L} is solvable if $\mathfrak{L}^{(m)} = 0$ for some integer m .*

9.2. **Examples of nilpotent algebras.** We already have examples of both nilpotent and solvable algebras. They are subalgebras of a simple Lie algebra. From our examples A_ℓ , B_ℓ , C_ℓ and D_ℓ we know that the basis of an algebra may be separated into three sets:

$$(168) \quad \mathfrak{B}_+ = \{E_\alpha \mid \alpha \in \Phi_+\}, \quad \mathfrak{B}_- = \{E_\alpha \mid \alpha \in \Phi_-\},$$

and the Cartan subalgebra \mathfrak{H} : a subalgebra of commutative semi-simple elements. Subalgebras \mathfrak{B}_\pm are known as Borel subalgebras.

For any $\alpha, \beta \in \Phi_+$ we have

$$(169) \quad [E_\alpha, E_\beta] = \begin{cases} cE_{\alpha+\beta}, & \alpha + \beta \in \Phi_+, \\ 0 & \text{otherwise} \end{cases}$$

The root system is finite (the algebra is finite), therefore there is a root $\alpha_0 \in \Phi_+$ such that $\alpha_0 + \alpha \notin \Phi_+$ for any other $\alpha \in \Phi_+$. This root is called the maximal root. It follows, the Borel subalgebra \mathfrak{B}_+ is nilpotent:

$$(170) \quad \underbrace{[\mathfrak{B}_+, [\mathfrak{B}_+, \dots [\mathfrak{B}_+, \mathfrak{B}_+] \dots]]}_{\text{several times}} = 0.$$

From (169), $[E_{\alpha_0}, E_\alpha] = 0$, therefore the maximal root vector is the center of \mathfrak{B}_+ (see definition 6). Existence of the center is the common feature of all nilpotent algebras: if $\mathfrak{L}^m = 0$ but $\mathfrak{L}^{m-1} \neq 0$, then $\mathfrak{L}^{m-1} = Z(\mathfrak{L})$.

One more definition and theorem (relating nilpotent algebras and nilpotent endomorphisms):

Definition 33. *An element x of Lie algebra is called ad-nilpotent if adx is nilpotent,*

$$(171) \quad (adx)^m \mathfrak{L} \stackrel{\text{def}}{=} \underbrace{[x, [x, \dots [x, \mathfrak{L}] \dots]]}_{m \text{ times}} = 0.$$

Theorem 8. (Engel) *A Lie algebra is nilpotent if and only if all its elements are ad-nilpotent.*

Proof is quite evident, use the definitions and finiteness of Lie algebra's dimension. \square

Remember now the Jordan-Chevalley theorem 5: if an element of simple algebra is nilpotent in one representation, it is nilpotent in all other representations. In this section we are talking not about a single element, but about the set of elements, whole (sub)algebras.

Comment: formally, an Abelian algebra is nilpotent. Feature of an Abelian algebra \mathfrak{A} is that $\mathfrak{A}^1 = 0$, i.e. $\mathfrak{A} = Z(\mathfrak{A})$ and $\text{ad}\mathfrak{A} \equiv 0$. We will exclude the Abelian subalgebras from the definition of nilpotent algebras.

9.3. Solvable algebras. Turn to the solvable subalgebras. Let $\mathfrak{C} = \mathfrak{B}_+ \oplus \mathfrak{H}$. We may see,

$$(172) \quad [\mathfrak{C}, \mathfrak{C}] = [\mathfrak{B}_+, \mathfrak{B}_+] \oplus [\mathfrak{H}, \mathfrak{B}_+] = \mathfrak{B}_+$$

i.e. the first derived algebra of \mathfrak{C} is the nilpotent algebra. The subsequent derived algebras will soon disappear, therefore \mathfrak{C} is the solvable algebra.

It appeared in this example, the first derived algebra of a solvable algebra is the nilpotent one. This is the subject of the criterium theorem:

Theorem 9. *A Lie algebra \mathfrak{C} is solvable if and only if the first term $[\mathfrak{C}, \mathfrak{C}]$ in the derived series is a nilpotent algebra.*

9.4. Representations of solvable algebras.

Theorem 10. (Lie) *Let V be a module for solvable algebra \mathfrak{C} . Then there exists a basis in V relative to which all the matrices of \mathfrak{C} are upper-triangular.*

The theorem states that there is a basis in V such that for any $x \in \mathfrak{C}$

$$(173) \quad \pi(x) = \begin{pmatrix} x_{11} & x_{22} & x_{33} & \dots \\ 0 & x_{22} & x_{23} & \dots \\ 0 & 0 & x_{33} & \dots \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

In particular, the nilpotent algebra $[\mathfrak{C}, \mathfrak{C}]$ is represented by strictly upper-triangular, and therefore nilpotent matrices. It proves the theorem 9.

The key feature of (173) is that the vector $\text{column}(1, 0, 0, \dots, 0)$ is a common eigenvector for all elements of the algebra.

Volunteers may read the proof of the theorem as a homework.

To prove Th. 10, we need a lemma.

Lemma 12. *Let $\mathfrak{C} \subset gl(V)$ be a solvable Lie algebra. Then there is a common eigenvector $v \in V$: $Av = \lambda(A)v$ for all $A \in \mathfrak{C}$.*

The eigenvalue $\lambda(A)$ in this lemma is some linear function on the algebra: if $Av = \lambda(A)v$ and $Bv = \lambda(B)v$ then $(A + B)v = (\lambda(A) + \lambda(B))v = \lambda(A + B)v$.

Proof.

(i) Consider first an algebra of the type $\mathfrak{L} = X + \mathfrak{J}$, where \mathfrak{J} is an ideal of \mathfrak{L} such that $\dim \mathfrak{L} - \dim \mathfrak{J} = 1$. X is a single element completing \mathfrak{J} to \mathfrak{L} , and since \mathfrak{J} is the ideal, $[X\mathfrak{J}] \subseteq \mathfrak{J}$.

Suppose \mathfrak{J} has a common eigenvector u , $Au = \lambda(A)u$ for any $A \in \mathfrak{J}$. Consider the of vectors

$$(174) \quad u, Xu, X^2u, \dots, X^{m-1}u,$$

and let us cut this chain on the first $X^m u$ such that

$$(175) \quad X^m u = \sum_{i=0}^{m-1} p_i X^i u.$$

It means, the vectors (174) form some basis of the invariant subspace of X . Since V is finite dimensional, basis (174) is finite.

Using

$$(176) \quad AX = XA + [AX], \quad A \text{ and } [AX] \in \mathfrak{J},$$

one may conclude

$$(177) \quad \begin{aligned} AXu &= \lambda(A)Xu + \lambda([AX])u, \\ \dots \\ AX^j u &= \lambda(A)X^j u + \sum_{k=0}^{j-1} \lambda_{jk}(A)X^k u, \end{aligned}$$

Therefore, in the subspace (174),

$$(178) \quad A = \begin{pmatrix} \lambda & * & \dots & * \\ & \lambda & \dots & * \\ & & \ddots & * \\ & & & \lambda \end{pmatrix}, \quad X = \begin{pmatrix} 0 & & & p_0 \\ 1 & 0 & & p_1 \\ & 1 & 0 & p_2 \\ & & \ddots & \ddots \\ & & & 1 & p_{m-1} \end{pmatrix},$$

where $*$ stand for λ_{jk} . Condition $[AX] \in \mathfrak{J}$ means that $[AX]$ must have the same upper-triangular form as A with diagonal $\lambda([AX])$. Direct calculation of the commutator of matrices (178) gives immediately $\lambda([AX]) = 0$ and all other $\lambda_{jk} = 0$ as well. Now let v be an eigenvector of X in the subspace (174), $Xv = cv$. Since A is the $\lambda(A)$ -multiple of unity, $Av = \lambda(A)v$. Denoting $c = \lambda(X)$, we have $Av = \lambda(A)v$ for any $A \in \mathfrak{L}$. We thus prove: *If an ideal of an algebra has a common eigenvector, then the algebra has a common eigenvector as well.*

(ii) Turn to a solvable algebra. Algebra is solvable if $\mathfrak{C}' \neq \mathfrak{C}$, i.e. $\dim \mathfrak{C} - \dim \mathfrak{C}' > 0$. Clearly, any subalgebra \mathfrak{J} : $\mathfrak{C}' \subset \mathfrak{J}$, $\mathfrak{J} \neq \mathfrak{C}$ is an ideal of \mathfrak{C} . Therefore a solvable algebra \mathfrak{C} may be presented as a chain

$$(179) \quad \mathfrak{C} = x_1 \oplus \mathfrak{C}_1, \quad \mathfrak{C}_1 = x_2 \oplus \mathfrak{C}_2, \dots$$

with some elements $x_k \in \mathfrak{C}$, so that \mathfrak{C}_{k+1} is an ideal of \mathfrak{C}_k . Thus

$$(180) \quad \mathfrak{C} = x_1 \oplus x_2 \oplus \dots \oplus x_{L-1} \oplus x_L$$

Here $L = \dim \mathfrak{C}$. Now use (i). The last element x_L has an eigenvector \Rightarrow the algebra $\mathfrak{C}_{L-2} = x_{L-1} \oplus x_L$ has a common eigenvector since $\mathfrak{C}_{L-1} = x_L$ is the ideal of $\mathfrak{C}_{L-2} \Rightarrow$ and so

on L times, any \mathfrak{C}_k has a common eigenvector, therefore the solvable algebra \mathfrak{C} has a common eigenvector. \square

Proof of Theorem 10. Lemma 12 is the tool for construction of the desired basis step-by-step. For given solvable Lie algebra \mathfrak{C} let v_1 be its common eigenvector for representation $\pi(\mathfrak{C})$, and v_2, \dots, v_n be a arbitrary set of vectors such that v_1, v_2, \dots, v_n is the basis of V . Then any $\pi(x)$ has the column of zeros below x_{11} in (173). Let V' be the vector space spanned by v_2, \dots, v_n , we have

$$(181) \quad \pi(x)V' = \pi'(x)V' + \lambda(x)v_1$$

where π' is defined by the choice of the basis v_2, \dots, v_n . Clearly, π' is the representation of \mathfrak{C} in V' and therefore we can choose v_2 to be the common eigenvector for all $\pi'(\mathfrak{C})$. It gives us the column of zeros below x_{22} in (173). Continuing this procedure, we restore the whole upper-triangular form (173). \square

10. STRUCTURE THEORY OF SEMI-SIMPLE LIE ALGEBRAS

10.1. Adjoint representation again. The adjoint representation introduces a natural symmetric bilinear form which plays an important role in the structure theory of Lie algebras: the Killing form defined by 20, page 13. Note its associativity property,

$$(182) \quad K([xy], z) = \text{Trace} \left([(\text{ad}x)(\text{ad}y) - (\text{ad}y)(\text{ad}x)](\text{ad}z) \right) = K(x, [yz]).$$

Definition 34. *Symmetric bilinear form $K(x, y)$ over a linear space \mathfrak{L} is said to be non-degenerate if there are no non-zero elements x with the property $K(x, \mathfrak{L}) = 0$.*

Lemma 13. *The set $\{x \in \mathfrak{L} \mid K(x, \mathfrak{L}) = 0\}$ is an ideal of \mathfrak{L} .*

Proof. If $K(x_1, \mathfrak{L}) = K(x_2, \mathfrak{L}) = 0$, then $K([x_1, x_2], \mathfrak{L}) = K(x_1, [x_2, \mathfrak{L}]) = 0$. \square

Lemma 14. *Let \mathfrak{J} be an ideal of \mathfrak{L} . Then the restriction of the Killing form to \mathfrak{J} coincides with the Killing form defined for \mathfrak{J} as an algebra in its own right.*

Proof. We must prove that

$$(183) \quad \text{Trace}_{\mathfrak{L}}(\text{ad}x \text{ ad}y) = \text{Trace}_{\mathfrak{J}}(\text{ad}x \text{ ad}y), \quad \forall x, y \in \mathfrak{J}.$$

Choose a basis in \mathfrak{J} and extend it to \mathfrak{L} . In this basis the matrices $\text{ad}x$, $\text{ad}y$ and $\text{ad}x \text{ ad}y$ have the block form

$$(184) \quad \text{ad}x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}, \quad \text{ad}y = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}, \quad \text{ad}x \text{ ad}y = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ 0 & 0 \end{pmatrix}.$$

so the contribution to the trace comes only from the action in \mathfrak{J} . \square

Theorem 11. (Cartan's criterion). *A Lie algebra is solvable if $K([\mathfrak{L}\mathfrak{L}], \mathfrak{L}) = 0$.*

Proof. According to the Lie theorem, in any representation we may choose a basis such that the solvable algebra may be represented by upper-triangular matrices. Adjoint representation is not an exception. If $x \in \text{ad}[\mathfrak{L}\mathfrak{L}]$ – strictly upper-triangular and $y \in \text{ad}\mathfrak{L}$ – upper triangular, it is easy to see that xy is strictly upper-triangular (has empty diagonal), therefore $K([\mathfrak{L}\mathfrak{L}], \mathfrak{L}) = 0$.

\square

Note, Cartan's criterion works not only in adjoint representation, but in any representation [with $K_{\pi}(x, y) = \text{Trace}(\pi(x)\pi(y))$].

Sometimes it is strange to see that very similar things are proven in theorems with different names. History of XIX century is mysterious.

10.2. Semi-simple Lie algebras.

Definition 35. (Semi-simple Lie algebra). A Lie algebra \mathfrak{L} is semi-simple if and only if it has no solvable ideal.

Do not mix semi-simple endomorphisms and semi-simple algebras. Evidently, simple algebra is semi-simple.

Theorem 12. A Lie algebra \mathfrak{L} is semi-simple if and only if its Killing form is non-degenerate.

Proof. If \mathfrak{L} has a solvable ideal, the Killing form is degenerative – it follows from lemma 14 and theorem 11. If Killing form is degenerative, i.e. there exist an element $x: K(x, \mathfrak{L}) = 0$, then x forms an ideal. One-dimensional ideal is of course solvable (easy to prove that one-dimensional ideals are centers of solvable ideals). □

Theorem 13. Let \mathfrak{L} be a semi-simple Lie algebra. Then there exist ideals $\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_k$ of \mathfrak{L} which are simple (as Lie algebras) such that

$$(185) \quad \mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \dots \oplus \mathfrak{L}_k .$$

The decomposition is unique to within numbering.

Proof. If \mathfrak{L} is not simple, then one may choose an arbitrary ideal \mathfrak{J} . Let \mathfrak{J}^\perp be the set of elements orthogonal to \mathfrak{J} under the Killing form. The Killing form must be zero on the intersection of \mathfrak{J} and \mathfrak{J}^\perp , $K(\mathfrak{J} \cap \mathfrak{J}^\perp, \mathfrak{L}) = 0$, so $\mathfrak{J} \cap \mathfrak{J}^\perp$ must be solvable, and hence must be zero. Iterating the process of decompositions \mathfrak{J} and \mathfrak{J}^\perp , we obtain the desired decomposition into simple ideals. □

An example of semi-simple Lie algebra was given in some of tutorials: $D_2 = A_1 \oplus A_1$.

10.3. Structure theory: Cartan (root space) decomposition. Now we show in three steps that any semi-simple algebra may be written as

$$(186) \quad \mathfrak{L} = \underbrace{\{E_\alpha \mid \alpha \in \Phi_+\}}_{\mathfrak{B}_+} \oplus \mathfrak{H} \oplus \underbrace{\{E_{-\alpha} \mid \alpha \in \Phi_+\}}_{\mathfrak{B}_-}, \quad [H, E_\alpha] = \alpha(H)E_\alpha ,$$

where \mathfrak{h} is the maximal commutative subalgebra (Cartan subalgebra), and \mathfrak{B}_\pm are two nilpotent subalgebras.

Step 1 Let x_i , $i = 1, \dots, n$, be some initial basis of the algebra, and we are considering the adjoint representation. There exists a simple computational procedure allowing one to construct step-by step a set of linearly independent elements $E_j = \sum_{i=1}^n k_{ji}x_i$, $j = 1, \dots, m < n$, such that

$$(187) \quad \det(\lambda - \sum_{j=1}^m c_j \text{ad}E_j) \equiv \lambda^n, \quad \forall c_1, \dots, c_m$$

(just considering characteristic polynomial for an arbitrary element of $\text{ad}\mathfrak{L}$). Maximal m is defined by condition: any other element $\text{ad}E_{m+1}$ satisfying (187) is a linear combination of $\{\text{ad}E_1, \dots, \text{ad}E_m\} = \mathfrak{B}$. Evidently, \mathfrak{B} is the subalgebra of nilpotent matrices (equation (187) is the condition for nilpotency of a matrix). Of course, construction of \mathfrak{B} is not unique, but all possible choices of \mathfrak{B} are equivalent.

Step 2 Let $\mathfrak{C} \in \mathfrak{L}$ be a set such that $K(\mathfrak{C}, \mathfrak{B}) = 0$. One may show (*lemma*), $[\mathfrak{C}, \mathfrak{C}] = \mathfrak{B}$, therefore \mathfrak{C} is solvable. Use the Lie theorem: we may choose the basis such that $\text{ad}\mathfrak{C}$ is upper-triangular. Use the Jordan-Chevalley theorem: any element may be decomposed into nilpotent part $\in \mathfrak{B}$ and semi-simple part – the diagonal one. Define $\text{ad}\mathfrak{h} = \text{diag}(\text{ad}\mathfrak{C})$.

We consider the adjoint representation, $\text{ad}x \cdot y = [xy]$, with equivalence: $[x = \text{a vector in } \mathfrak{L}] \Leftrightarrow [\text{ad}x = \text{the endomorphism of } \mathfrak{L}]$. Since $\text{ad}\mathfrak{h}$ by construction is diagonal, $\text{ad}H \cdot E_j = [H, E_j] \simeq E_j$, therefore our construction gives the root system: $\mathfrak{B} = \sum_{\alpha} E_{\alpha}$, $[H, E_{\alpha}] = \alpha(H)E_{\alpha}$, $\alpha \in \Phi_+$. Actually, this is the definition of Φ_+ as the set of indices of elements of \mathfrak{B} .

Step 3 The basis $\{E_{-\alpha}\}$ is defined by conditions $K(E_{-\alpha}, E_{\beta}) = \delta_{\alpha, \beta}$, $K(E_{-\alpha}, \mathfrak{h}) = 0$ and $K(E_{-\alpha}, E_{-\beta}) = 0$. The index $-\alpha$ is self-consistent, due to the associativity (182),

$$(188) \quad \alpha(H) = K([H, E_{\alpha}], E_{-\alpha}) = -K([H, E_{-\alpha}], E_{\alpha}) \Rightarrow [H, E_{-\alpha}] = -\alpha(H)E_{-\alpha}.$$

Using again the the associativity (182), we can see that $K([E_{\alpha}E_{-\alpha}], E_{\pm\beta}) = 0$, therefore $[E_{\alpha}, E_{-\alpha}] \in \mathfrak{h}$. The complete root system $\Phi = \Phi_+ \cup \Phi_-$.

It is easy to show that (186) exhaust the whole \mathfrak{L} (*lemma*), otherwise K is degenerative.

Using the root decomposition, we may find in particular

$$(189) \quad K(H, H') = \sum_{\alpha \in \Phi} \alpha(H)\alpha(H'), \quad H, H' \in \mathfrak{h}.$$

An element $H \in \mathfrak{h}$ is a *regular element* if $\alpha(H) \neq 0$ for all $\alpha \in \Phi$. Regular elements do exist (otherwise, K is degenerative), therefore \mathfrak{h} is not empty. A feature of a regular element H is that $[H, X] = 0 \Leftrightarrow X \in \mathfrak{h}$.

We know that the root system Φ is a lattice in the dual space \mathfrak{h}^* . Killing form allows one to establish an isomorphism between \mathfrak{h} and \mathfrak{h}^* :

$$(190) \quad \lambda \in \mathfrak{h}^* \Leftrightarrow h_\lambda \in \mathfrak{h} : \lambda(H) = K(h_\lambda, H).$$

10.4. Example of steps 1-3. (Home reading for volunteers). Let me show how the steps 1-3 work for A_1 . We know that the matrices (22) define the adjoint representation of A_1 . An element of the algebra in adjoint representation is

$$(191) \quad X = x_1T_1 + x_2T_2 + x_3T_3 = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}.$$

(We have changed the sign of T_2 from (22). This is not essential, but makes the adjoint representation more nice.) The Killing form in the basis T_j is

$$(192) \quad K(X, Y) = \text{Trace} \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} = -2(x_1y_1 + x_2y_2 + x_3y_3).$$

Calculate now the characteristic polynomial,

$$(193) \quad \det(\lambda - X) = \lambda^3 + (x_1^2 + x_2^2 + x_3^2)\lambda$$

Element X is nilpotent if $x_1^2 + x_2^2 + x_3^2 = 0$. Choose a simple solution of this equation, e.g. $x_1 = 1, x_2 = i$, and define $E = T_1 + iT_2$. One may verify in the matrix form, $E^3 = 0$.

Question: may we choose another nilpotent Y such that $cE + Y$ is nilpotent (for any c) as well? Verify: $\det(\lambda - cE - Y) = \lambda^3 + ((c + y_1)^2 + (ic + y_2)^2 + y_3^2)\lambda$, therefore $cE + Y$ is nilpotent if $2c(y_1 + iy_2) + (y_1^2 + y_2^2 + y_3^2) = 0 \Rightarrow y_1 + iy_2 = 0$ and $y_3 = 0$. Thus $Y = y_1E$, and cE is the maximal nilpotent subalgebra.

Construct now \mathfrak{C} : $C = c_1T_1 + c_2T_2 + c_3T_3 \in \mathfrak{C}$ if $K(C, E) = 0$.

$$(194) \quad \text{Trace}(CE) = -2(c_1 + ic_2) = 0$$

therefore $C = c_1E + c_3T_3 \in \mathfrak{C}$. Consider now the basis (the Lie theorem basis!)

$$(195) \quad v_1 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

In this basis

$$(196) \quad Cv_1 = ic_3v_1, \quad Cv_2 = -ic_1v_1, \quad Cv_3 = -ic_3v_3 + 2ic_1v_2,$$

or, in matrix form

$$(197) \quad C = c_1E + c_3T_3 = \begin{pmatrix} 0 & c_3 & -ic_1 \\ -c_3 & 0 & c_1 \\ ic_1 & -c_1 & 0 \end{pmatrix}_{\text{initial basis}} = \begin{pmatrix} ic_3 & -ic_1 & 0 \\ 0 & 0 & 2ic_1 \\ 0 & 0 & -ic_3 \end{pmatrix}_{\text{basis } v_1, v_2, v_3}$$

It is clear in the basis (195) that $H = -2iT_3$ is diagonal and E is nilpotent.

Last step is to construct F : $K(F, E) = 1$, $K(F, H) = 0$ and $K(F, F) = 0$. It gives uniquely $F = -(T_1 - iT_2)/4$. Verification:

$$(198) \quad [HE] = 2E, \quad [HF] = 2F, \quad [EF] = -\frac{1}{4}H.$$

Standard basis corresponds to $e = E$ and $f = -4F$.

Conclusion: just to note the fundamental feature of adjoint representation: any element x of algebra corresponds simultaneously to a vector v_x and to a matrix adx . This correspondence in the initial basis was

$$(199) \quad T_1 \leftrightarrow t_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T_2 \leftrightarrow t_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T_3 \leftrightarrow t_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Compare this with the basis (195):

$$(200) \quad v_1 \leftrightarrow E, \quad v_2 \leftrightarrow H, \quad v_3 \leftrightarrow F.$$

It follows, the set of $E_{-\alpha}$ appears in fact at the stage of Lie theorem: the Lie basis must have the form

$$(201) \quad \{v_\alpha\}, \{v_H\}, \{v_{-\alpha}\}.$$

ASSIGNMENT 1

Select five

1. Shew that a Lie algebra is Abelian if and only if its every subspace is an ideal.
2. Let the Lie algebra \mathfrak{L} is spanned by

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find an ideal of \mathfrak{L} .

3. Shew the the intersection of two ideals \mathfrak{I} and \mathfrak{I}' of a Lie algebra \mathfrak{L} is an ideal.
4. For a Lie algebra \mathfrak{L} we define the derived series inductively by $\mathfrak{L}^{(k+1)} = [\mathfrak{L}^{(k)}, \mathfrak{L}^{(k)}]$, $\mathfrak{L}^{(0)} = \mathfrak{L}$. Shew that each $\mathfrak{L}^{(k)}$ is an ideal of all $\mathfrak{L}^{(k')}$ for $k' < k$.
5. Let 3×3 matrices X and Z are defined by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Shew that the set $\mathfrak{S} = \{X, Z\}$ is irreducible.

6. Algebra $o(n)$ was introduced as the algebra of antisymmetric matrices. It has the basis $E_{[ij]} = E_{ij} - E_{ji}$, $i, j = 1, \dots, n$. Find the structure constants in this basis

$$[E_{[ij]}, E_{[kl]}] = \dots ?$$

7. Clifford Algebra \mathcal{C}_n is the associative algebra with generators γ_j satisfying

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}, \quad i, j = 1, \dots, n.$$

Shew that

$$\pi : E_{[ij]} \mapsto \frac{1}{4}(\gamma_i \gamma_j - \gamma_j \gamma_i),$$

where $E_{[ij]}$ where defined in the previous problem, is the homomorphism of $o(n)$.

8. Let n be even, $o(n) = D_{n/2}$. Find the Cartan subalgebra and root vectors in the $o(n)$ basis $E_{[ij]}$.

A comment for 8. $x' = (p, q, r)$,

$$x = p \otimes \frac{1 + \sigma_2}{2} = p^T \otimes \frac{1 - \sigma_2}{2} + r \otimes \frac{\sigma_3 - i\sigma_1}{2} + q \otimes \frac{\sigma_3 + i\sigma_1}{2}$$

Additional collection of problems.

- Let $XZ = \omega ZX$, X and Z are $n \times n$ matrices such that $X^n = Z^n = 1$, $\omega = e^{2\pi i/n}$.

Shew that $\mathfrak{S} = \{X, Z\}$ is irreducible .

- Let A^2 is a semi-simple endomorphism. Shew that A is semi-simple.

- Shew that

$$h = -2i \frac{\partial}{\partial \phi}, \quad e = e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right), \quad f = e^{-i\phi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right).$$

is the realization of A_1 .

- For the previous problem, find $v_{\lambda=0}$ in the module with the highest weight $J = 2j$.
- Using the root systems, shew that $B_2 \simeq C_2$.