曲線座標系における微分演算子

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1 準備

F を n 階のテンソル $(n = 0, 1, 2, \cdots)$ とし、F の勾配を ∇F とかく。

$$\nabla F = \frac{\partial F}{\partial \boldsymbol{x}}. \tag{1}$$

基本ベクトルを $\check{\boldsymbol{e}}_i$(必ずしも単位ベクトルではない) とすると、勾配の成分は

$$\check{\boldsymbol{e}}_i \cdot \nabla F = \frac{\partial F}{\partial x_i} \quad (i = 1, 2, 3) \tag{2}$$

と書ける。F をスカラー f とすると

$$(\nabla f)_i = \check{\boldsymbol{e}}_i \cdot \nabla f = \frac{\partial f}{\partial x_i}. \tag{3}$$

F をベクトル $\boldsymbol{g}$ とすると、$\nabla F = \nabla \boldsymbol{g}$ は 2 階のテンソルでありその ij 成分は

$$(\nabla \boldsymbol{g})_{ij} = \check{\boldsymbol{e}}_i \cdot \nabla \boldsymbol{g} \cdot \check{\boldsymbol{e}}_j = \frac{\partial \boldsymbol{g}}{\partial x_i} \cdot \check{\boldsymbol{e}}_j = \frac{\partial g_j}{\partial x_i}. \tag{4}$$

成分行列は

$$(\nabla \boldsymbol{g})_{ij} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_3}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_3}{\partial x_2} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} & \frac{\partial g_3}{\partial x_3} \end{pmatrix}. \tag{5}$$

微分する変数 x_i の添字 i が行、微分されるベクトルの成分 g_j の添字 j が列になっている。

F が 2 階のテンソル $\overleftrightarrow{T}$ であるとする。$\overleftrightarrow{T}$ の勾配は 3 階のテンソルであり、その成分を $(\nabla \overleftrightarrow{T})_{ijk}$ とすると、

$$\nabla \overleftrightarrow{T} = (\nabla \overleftrightarrow{T})_{ijk} \check{\boldsymbol{e}}_i \check{\boldsymbol{e}}_j \check{\boldsymbol{e}}_k. \tag{6}$$

のように triads であらわされる。成分を求めるには triads $\check{\boldsymbol{e}}_i \check{\boldsymbol{e}}_j \check{\boldsymbol{e}}_k$ を右から、または左から内積することによって得られる。

$$\begin{aligned} (\nabla \overleftrightarrow{T})_{ijk} &= ((\nabla \overleftrightarrow{T} \cdot \check{\boldsymbol{e}}_k) \cdot \check{\boldsymbol{e}}_j) \cdot \check{\boldsymbol{e}}_i = \check{\boldsymbol{e}}_k \cdot (\check{\boldsymbol{e}}_j \cdot (\check{\boldsymbol{e}}_i \cdot \nabla \overleftrightarrow{T})) \\ &= \check{\boldsymbol{e}}_k \cdot \left(\check{\boldsymbol{e}}_j \cdot \left(\frac{\partial \overleftrightarrow{T}}{\partial x_i} \right) \right) = \frac{\partial (\check{\boldsymbol{e}}_k \cdot \overleftrightarrow{T} \cdot \check{\boldsymbol{e}}_j)}{\partial x_i} = \frac{\partial T_{jk}}{\partial x_i}. \end{aligned} \tag{7}$$

2　円柱座標系

デカルト座標系 $\boldsymbol{x} = (x, y, z)$ から円柱座標系 $\boldsymbol{x}' = (r, \theta, z)$ への変換 $\mathcal{F}_{\rm c} : \boldsymbol{x} \to \boldsymbol{x}'$ は

$$\left\{ \begin{array}{ccl} x &=& r\cos\theta \\ y &=& r\sin\theta \\ z &=& z \end{array} \right. \tag{8}$$

で定義される。

デカルト座標系における勾配 ∇F と円柱座標系における F の勾配 $\nabla' F$ は

$$\nabla F = \frac{\partial F}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}} \cdot \frac{\partial F}{\partial \boldsymbol{x}'} = \nabla \boldsymbol{x}' \cdot \nabla' F \tag{9}$$

で関係づけられる。座標変換 $\mathcal{F}_{\rm c}$ のヤコビアン $J_{\mathcal{F}_{\rm c}}$ は

$$J_{\mathcal{F}_{\rm c}} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |J_{\mathcal{F}_{\rm c}}| = \frac{1}{r} \tag{10}$$

で定義される。$J_{\mathcal{F}_{\rm c}} = ^t(\partial \boldsymbol{x}'/\partial \boldsymbol{x})$ である。$J_{\mathcal{F}_{\rm c}}$ の各行は、各座標の勾配 ∇r, $\nabla \theta$, ∇z であり、座標系の反変基本ベクトルとよばれる。$|\nabla r| = 1$, $|\nabla \theta| = 1/r$, $|\nabla z| = 1$ であるから、基本単位ベクトル $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ との間に、

$$\boldsymbol{e}_r = \nabla r, \quad \boldsymbol{e}_\theta = r\nabla\theta, \quad \boldsymbol{e}_z = \nabla z \tag{11}$$

なる関係がある。一方、$\nabla' F = \partial \boldsymbol{x}/\partial \boldsymbol{x}' \nabla F = ^t J_{\mathcal{F}_{\rm c}}^{-1} \nabla F$ であり、陽に書き下すと

$$\frac{\partial F}{\partial r} = \frac{\partial \boldsymbol{x}}{\partial r} \cdot \frac{\partial F}{\partial \boldsymbol{x}}, \quad \frac{\partial F}{\partial \theta} = \frac{\partial \boldsymbol{x}}{\partial \theta} \cdot \frac{\partial F}{\partial \boldsymbol{x}}, \quad \frac{\partial F}{\partial z} = \frac{\partial \boldsymbol{x}}{\partial z} \cdot \frac{\partial F}{\partial \boldsymbol{x}} \tag{12}$$

となる。$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}')$ の偏微分係数は

$$J_{\mathcal{F}_{\rm c}}^{-1} = J_{\mathcal{F}_{\rm c}^{-1}} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |J_{\mathcal{F}_{\rm c}^{-1}}| = r \tag{13}$$

より

$$\frac{\partial \boldsymbol{x}}{\partial r} = \boldsymbol{e}_r, \quad \frac{\partial \boldsymbol{x}}{\partial \theta} = r\boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{x}}{\partial z} = \boldsymbol{e}_z. \tag{14}$$

これらは、共変基本ベクトルとよばれる。

円柱座標系においては、基本単位ベクトルは θ に沿って向きを変えるため定ベクトルではない。各偏微分係数をもとめると、

$$\begin{array}{lll} \frac{\partial \boldsymbol{e}_r}{\partial r} = 0 & \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta & \frac{\partial \boldsymbol{e}_r}{\partial z} = 0 \\ \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 & \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r & \frac{\partial \boldsymbol{e}_\theta}{\partial z} = 0 \\ \frac{\partial \boldsymbol{e}_z}{\partial r} = 0 & \frac{\partial \boldsymbol{e}_z}{\partial \theta} = 0 & \frac{\partial \boldsymbol{e}_z}{\partial z} = 0. \end{array} \tag{15}$$

勾配 (gradient)

スカラー f の勾配は、反変ベクトルを用いて

$$\nabla f = \nabla r \frac{\partial f}{\partial r} + \nabla \theta \frac{\partial f}{\partial \theta} + \nabla z \frac{\partial f}{\partial z} = \boldsymbol{e}_r \frac{\partial f}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \boldsymbol{e}_z \frac{\partial f}{\partial z} \tag{16}$$

と求められる。または、共変ベクトルを用いると、

$$\frac{\partial f}{\partial r} = \boldsymbol{e}_r \cdot \nabla f \tag{17}$$

より

$$(\nabla f)_r = \boldsymbol{e}_r \cdot \nabla f = \frac{\partial f}{\partial r}. \tag{18}$$

同様にして、

$$(\nabla f)_\theta = \boldsymbol{e}_\theta \cdot \nabla f = \frac{1}{r} \frac{\partial f}{\partial \theta}, \tag{19}$$

$$(\nabla f)_z = \boldsymbol{e}_z \cdot \nabla f = \frac{\partial f}{\partial z}. \tag{20}$$

次に、基本単位ベクトルの勾配テンソルを計算する。単位テンソル $\overleftrightarrow{I} = \boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_z\boldsymbol{e}_z$ を用いると、

$$\nabla \boldsymbol{e}_r = \overleftrightarrow{I} \cdot \nabla \boldsymbol{e}_r = (\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_z\boldsymbol{e}_z) \cdot \nabla \boldsymbol{e}_r = \boldsymbol{e}_r \frac{\partial \boldsymbol{e}_r}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial \boldsymbol{e}_r}{\partial \theta} + \boldsymbol{e}_z \frac{\partial \boldsymbol{e}_r}{\partial z} = \frac{1}{r} \boldsymbol{e}_\theta \boldsymbol{e}_\theta. \tag{21}$$

同様に、

$$\nabla \boldsymbol{e}_\theta = -\frac{1}{r} \boldsymbol{e}_\theta \boldsymbol{e}_r, \tag{22}$$

$$\nabla \boldsymbol{e}_z = 0. \tag{23}$$

任意のベクトル $\boldsymbol{g}$ の成分が $\boldsymbol{g} = \boldsymbol{e}_r g_r + \boldsymbol{e}_\theta g_\theta + \boldsymbol{e}_z g_z$ と書けているとすると、$\boldsymbol{g}$ の勾配テンソルは

$$\begin{aligned}
\nabla \boldsymbol{g} &= \nabla(\boldsymbol{e}_r g_r + \boldsymbol{e}_\theta g_\theta + \boldsymbol{e}_z g_z) \\
&= \nabla g_r \boldsymbol{e}_r + \nabla g_\theta \boldsymbol{e}_\theta + \nabla g_z \boldsymbol{e}_z + g_r \nabla \boldsymbol{e}_r + g_\theta \nabla \boldsymbol{e}_\theta \\
&= \left(\boldsymbol{e}_r \frac{\partial g_r}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial g_r}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_r}{\partial z} \right) \boldsymbol{e}_r + \left(\boldsymbol{e}_r \frac{\partial g_\theta}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial g_\theta}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_\theta}{\partial z} \right) \boldsymbol{e}_\theta \\
&\quad + \left(\boldsymbol{e}_r \frac{\partial g_z}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial g_z}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_z}{\partial z} \right) \boldsymbol{e}_z + \frac{1}{r} \boldsymbol{e}_\theta \boldsymbol{e}_\theta g_r - \frac{1}{r} \boldsymbol{e}_\theta \boldsymbol{e}_r g_\theta \\
&= \begin{pmatrix} \frac{\partial g_r}{\partial r} & \frac{\partial g_\theta}{\partial r} & \frac{\partial g_z}{\partial r} \\ \frac{1}{r}\frac{\partial g_r}{\partial \theta} - \frac{g_\theta}{r} & \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{g_r}{r} & \frac{1}{r}\frac{\partial g_z}{\partial \theta} \\ \frac{\partial g_r}{\partial z} & \frac{\partial g_\theta}{\partial z} & \frac{\partial g_z}{\partial z} \end{pmatrix}
\end{aligned} \tag{24}$$

と計算できる。

ベクトル場の発散 (divergence) と回転 (rotation/curl)

ベクトル場 $\boldsymbol{g}$ の発散は

$$\nabla \cdot \boldsymbol{g} = \mathrm{Tr}(\nabla \boldsymbol{g}) = \boldsymbol{e}_i \cdot \nabla \boldsymbol{g} \cdot \boldsymbol{e}_i = \frac{\partial g_i}{\partial x_i} \tag{25}$$

で与えられる。

基本単位ベクトルの発散を計算すると、

$$\nabla \cdot \boldsymbol{e}_r = \mathrm{Tr}\left(\frac{1}{r}\boldsymbol{e}_\theta \boldsymbol{e}_\theta\right) = \frac{1}{r}, \tag{26}$$

$$\nabla \cdot \boldsymbol{e}_\theta = 0, \quad \nabla \cdot \boldsymbol{e}_z = 0. \tag{27}$$

よって、

$$\begin{aligned} \nabla \cdot \boldsymbol{g} &= \nabla \cdot (\boldsymbol{e}_r g_r + \boldsymbol{e}_\theta g_\theta + \boldsymbol{e}_z g_z) \\ &= \boldsymbol{e}_r \cdot \nabla g_r + \boldsymbol{e}_\theta \cdot \nabla g_\theta + \boldsymbol{e}_z \cdot \nabla g_z + \frac{g_r}{r} \\ &= \frac{\partial g_r}{\partial r} + \frac{g_r}{r} + \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{\partial g_z}{\partial z}. \end{aligned} \tag{28}$$

ベクトル場 $\boldsymbol{g}$ の回転は

$$\nabla \times \boldsymbol{g} = \sum_{i=1}^{3} \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \nabla \boldsymbol{g} \cdot \boldsymbol{e}_k - \boldsymbol{e}_k \cdot \nabla \boldsymbol{g} \cdot \boldsymbol{e}_j) \tag{29}$$

で与えられる。基本単位ベクトルの回転は、(11) より

$$\nabla \times \boldsymbol{e}_r = 0, \quad \nabla \times \boldsymbol{e}_z = 0, \tag{30}$$

$$\nabla \times \boldsymbol{e}_\theta = \nabla r \times \nabla \theta = \boldsymbol{e}_r \times \frac{1}{r}\boldsymbol{e}_\theta = \frac{1}{r}\boldsymbol{e}_z. \tag{31}$$

よって、

$$\begin{aligned} \nabla \times \boldsymbol{g} &= \nabla \times (\boldsymbol{e}_r g_r + \boldsymbol{e}_\theta g_\theta + \boldsymbol{e}_z g_z) \\ &= \nabla g_r \times \boldsymbol{e}_r + \nabla g_\theta \times \boldsymbol{e}_\theta + \nabla g_z \times \boldsymbol{e}_z + \frac{g_\theta}{r}\boldsymbol{e}_z \\ &= \left(\boldsymbol{e}_r \frac{\partial g_r}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_r}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_r}{\partial z}\right) \times \boldsymbol{e}_r + \left(\boldsymbol{e}_r \frac{\partial g_\theta}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_\theta}{\partial z}\right) \times \boldsymbol{e}_\theta \\ &\quad + \left(\boldsymbol{e}_r \frac{\partial g_z}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_z}{\partial \theta} + \boldsymbol{e}_z \frac{\partial g_z}{\partial z}\right) \times \boldsymbol{e}_z + \frac{g_\theta}{r}\boldsymbol{e}_z \\ &= \left(\frac{1}{r}\frac{\partial g_z}{\partial \theta} - \frac{\partial g_\theta}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{\partial g_\theta}{\partial r} + \frac{g_\theta}{r} - \frac{1}{r}\frac{\partial g_r}{\partial \theta}\right)\boldsymbol{e}_z. \end{aligned} \tag{32}$$

ラプラシアン

2 階微分演算子ラプラシアン Δ は

$$\Delta = \nabla \cdot \nabla \tag{33}$$

で定義される。任意の座標系 $\boldsymbol{x}'$ においてスカラー場のラプラシアン Δf を計算すると

$$\begin{aligned}\Delta f &= \nabla\cdot(\nabla f) = \nabla\cdot\left(\frac{\partial f}{\partial x_i'}\nabla x_i'\right) = \left(\nabla\frac{\partial f}{\partial x_i'}\right)\cdot\nabla x_i' + \frac{\partial f}{\partial x_i'}\Delta x_i' \\ &= \frac{\partial^2 f}{\partial x_j'\partial x_i'}\nabla x_j'\cdot\nabla x_i' + \frac{\partial f}{\partial x_i'}\Delta x_i'. \end{aligned} \tag{34}$$

よって、ラプラシアンを計算するためには当該座標系における $\nabla x_i'$ および $\Delta x_i'$ が必要である。円柱座標系における座標の微分 $\nabla x_i'$, $\Delta x_i'$ を計算する。$\nabla x_i'$ は (11) で既に計算されている。ラプラシアンを計算すると

$$\Delta r = \nabla\cdot\boldsymbol{e}_r = \boldsymbol{e}_r\cdot\nabla 1 + \frac{1}{r} = \frac{1}{r}, \tag{35}$$

$$\Delta\theta = \nabla\cdot\left(\frac{\boldsymbol{e}_\theta}{r}\right) = \boldsymbol{e}_\theta\cdot\nabla\left(\frac{1}{r}\right) = \boldsymbol{e}_\theta\cdot\boldsymbol{e}_r\left(-\frac{1}{r^2}\right) = 0, \tag{36}$$

$$\Delta z = \nabla\cdot\boldsymbol{e}_z = 0. \tag{37}$$

以上より円柱座標系におけるラプラシアンは

$$\begin{aligned}\Delta f &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{r}\frac{\partial f}{\partial r} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \tag{38}$$

任意のベクトル場 $\boldsymbol{g}$ にラプラシアンを作用させるとき、ベクトル場の各成分にラプラシアンを作用するだけでは、正しい結果は得られない。なぜなら、座標成分にかかっている基本ベクトルが定ベクトルではないからである。$\Delta\boldsymbol{g}$ の各成分を計算する。$\boldsymbol{e}_i$ 方向成分は

$$(\Delta\boldsymbol{g})_i = \boldsymbol{e}_i\cdot\Delta\boldsymbol{g} = (\nabla\cdot\nabla\boldsymbol{g})\cdot\boldsymbol{e}_i. \tag{39}$$

ここで公式

$$\nabla\cdot(\overleftrightarrow{T}\cdot\boldsymbol{f}) = (\nabla\cdot\overleftrightarrow{T})\cdot\boldsymbol{f} + \mathrm{Tr}({}^t\overleftrightarrow{T}\cdot\nabla\boldsymbol{f}) \tag{40}$$

を用いる。証明は

$$\nabla\cdot(\overleftrightarrow{T}\cdot\boldsymbol{f}) = \mathrm{Tr}(\nabla(\overleftrightarrow{T}\cdot\boldsymbol{f})) = \mathrm{Tr}(\nabla\overleftrightarrow{T}\cdot\boldsymbol{f}) + \mathrm{Tr}({}^t\overleftrightarrow{T}\cdot\nabla\boldsymbol{f}) \tag{41}$$

より

$$\mathrm{Tr}(\nabla\overleftrightarrow{T}\cdot\boldsymbol{f}) = (\nabla\cdot\overleftrightarrow{T})\cdot\boldsymbol{f} \tag{42}$$

を示せばよい。

$$\nabla\overleftrightarrow{T}\cdot\boldsymbol{f} = \left(\frac{\partial T_{jk}}{\partial x_i}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\right)\cdot(f_l\boldsymbol{e}_l) = f_k\frac{\partial T_{jk}}{\partial x_i}\boldsymbol{e}_i\boldsymbol{e}_j \tag{43}$$

であり、縮約をとると

$$\mathrm{Tr}(\nabla\overleftrightarrow{T}\cdot\boldsymbol{f}) = f_k\frac{\partial T_{ik}}{\partial x_i}. \tag{44}$$

一方

$$\nabla\cdot\overleftrightarrow{T} = \frac{\partial T_{ij}}{\partial i}\boldsymbol{e}_j \tag{45}$$

より (42) が示された。(証明終わり)

上記公式を用いると、

$$(\nabla\cdot\nabla\boldsymbol{g})\cdot\boldsymbol{e}_i=\nabla\cdot(\nabla\boldsymbol{g}\cdot\boldsymbol{e}_i)-\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_i). \tag{46}$$

公式

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b})=(\nabla\boldsymbol{a})\cdot\boldsymbol{b}+(\nabla\boldsymbol{b})\cdot\boldsymbol{a} \tag{47}$$

より

$$\nabla\boldsymbol{g}\cdot\boldsymbol{e}_i=\nabla(\boldsymbol{g}\cdot\boldsymbol{e}_i)-\nabla\boldsymbol{e}_i\cdot\boldsymbol{g}. \tag{48}$$

(46) に代入すると、

$$(\Delta\boldsymbol{g})_i=\Delta g_i-\nabla\cdot(\nabla\boldsymbol{e}_i\cdot\boldsymbol{g})-\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_i). \tag{49}$$

右辺第 1 項はスカラーに対するラプラシアンであり、すでに求めたラプラシアンを $\boldsymbol{g}$ の各成分に作用させれば求まる。また、第 2, 3 項はすでに計算されている基本ベクトルの勾配を用いて計算できる。

各成分を計算する。

$$\nabla\cdot(\nabla\boldsymbol{e}_r\cdot\boldsymbol{g}) = \nabla\cdot\left(\left(\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta\right)\cdot(\boldsymbol{e}_r g_r+\boldsymbol{e}_\theta g_\theta+\boldsymbol{e}_z g_z)\right)=\nabla\cdot\left(\frac{g_\theta}{r}\boldsymbol{e}_\theta\right)=\frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}, \tag{50}$$

$$\nabla\cdot(\nabla\boldsymbol{e}_\theta\cdot\boldsymbol{g}) = \nabla\cdot\left(\left(-\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r\right)\cdot(\boldsymbol{e}_r g_r+\boldsymbol{e}_\theta g_\theta+\boldsymbol{e}_z g_z)\right)=-\nabla\cdot\left(\frac{g_r}{r}\boldsymbol{e}_\theta\right)=-\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}, \tag{51}$$

$$\nabla\cdot(\nabla\boldsymbol{e}_z\cdot\boldsymbol{g}) = 0. \tag{52}$$

また、

$${}^t(\nabla\boldsymbol{g})=\boldsymbol{e}_r\nabla g_r+\boldsymbol{e}_\theta\nabla g_\theta+\boldsymbol{e}_z\nabla g_z+\frac{g_r}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta-\frac{g_\theta}{r}\boldsymbol{e}_r\boldsymbol{e}_\theta \tag{53}$$

より

$$\begin{aligned}
\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_r) &= \mathrm{Tr}\left[\left(\boldsymbol{e}_r\nabla g_r+\boldsymbol{e}_\theta\nabla g_\theta+\boldsymbol{e}_z\nabla g_z+\frac{g_r}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta-\frac{g_\theta}{r}\boldsymbol{e}_r\boldsymbol{e}_\theta\right)\cdot\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta\right]\\
&= \frac{1}{r}\mathrm{Tr}\left(\boldsymbol{e}_r\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial g_r}{\partial\theta}+\boldsymbol{e}_\theta\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial g_\theta}{\partial\theta}+\boldsymbol{e}_z\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial g_z}{\partial\theta}+\boldsymbol{e}_\theta\boldsymbol{e}_\theta\frac{g_r}{r}-\boldsymbol{e}_r\boldsymbol{e}_\theta\frac{g_\theta}{r}\right)\\
&= \frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}+\frac{g_r}{r^2},
\end{aligned} \tag{54}$$

$$\begin{aligned}
\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_\theta) &= \mathrm{Tr}\left[\left(\boldsymbol{e}_r\nabla g_r+\boldsymbol{e}_\theta\nabla g_\theta+\boldsymbol{e}_z\nabla g_z+\frac{g_r}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta-\frac{g_\theta}{r}\boldsymbol{e}_r\boldsymbol{e}_\theta\right)\cdot\left(-\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r\right)\right]\\
&= -\frac{1}{r}\mathrm{Tr}\left(\boldsymbol{e}_r\boldsymbol{e}_r\frac{1}{r}\frac{\partial g_r}{\partial\theta}+\boldsymbol{e}_\theta\boldsymbol{e}_r\frac{1}{r}\frac{\partial g_\theta}{\partial\theta}+\boldsymbol{e}_z\boldsymbol{e}_r\frac{1}{r}\frac{\partial g_z}{\partial\theta}+\boldsymbol{e}_\theta\boldsymbol{e}_r\frac{g_r}{r}-\boldsymbol{e}_r\boldsymbol{e}_r\frac{g_\theta}{r}\right)\\
&= -\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}+\frac{g_\theta}{r^2},
\end{aligned} \tag{55}$$

$$\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_z) = 0. \tag{56}$$

以上をまとめると、

$$(\Delta g)_r = \Delta g_r-\frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}-\left(\frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}+\frac{g_r}{r^2}\right)=\Delta g_r-\frac{2}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{g_r}{r^2}, \tag{57}$$

$$(\Delta g)_\theta = \Delta g_\theta+\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}-\left(-\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}+\frac{g_\theta}{r^2}\right)=\Delta g_\theta+\frac{2}{r^2}\frac{\partial g_r}{\partial\theta}-\frac{g_\theta}{r^2}, \tag{58}$$

$$(\Delta g)_z = \Delta g_z \tag{59}$$

となる。

3　極座標

デカルト座標系 $\boldsymbol{x} = (x, y, z)$ から極座標系 $\boldsymbol{x}' = (r, \theta, \phi)$ への変換 $\mathcal{F}_\mathrm{p} : \boldsymbol{x} \to \boldsymbol{x}'$ は

$$\left\{ \begin{array}{lcl} x &=& r\sin\theta\cos\phi \\ y &=& r\sin\theta\sin\phi \\ z &=& r\cos\theta \end{array} \right. \tag{60}$$

で定義される。

ヤコビアン

$$J_{\mathcal{F}_\mathrm{p}} = \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\phi & -\frac{1}{r}\sin\theta \\ -\frac{1}{r}\frac{\sin\phi}{\sin\theta} & \frac{1}{r}\frac{\cos\phi}{\sin\theta} & 0 \end{pmatrix}, \quad |J_{\mathcal{F}_\mathrm{p}}| = \frac{1}{r^2\sin\theta} \tag{61}$$

$$J_{\mathcal{F}_\mathrm{p}^{-1}} = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}, \quad |J_{\mathcal{F}_\mathrm{p}^{-1}}| = r^2\sin\theta \tag{62}$$

基本ベクトル

$$\nabla r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \nabla\theta = \begin{pmatrix} \frac{1}{r}\cos\theta\cos\phi \\ \frac{1}{r}\cos\theta\sin\phi \\ -\frac{1}{r}\sin\theta \end{pmatrix}, \quad \nabla\phi = \begin{pmatrix} -\frac{1}{r}\frac{\sin\phi}{\sin\theta} \\ \frac{1}{r}\frac{\cos\phi}{\sin\theta} \\ 0 \end{pmatrix} \tag{63}$$

$$\boldsymbol{e}_r = \nabla r, \quad \boldsymbol{e}_\theta = r\nabla\theta, \quad \boldsymbol{e}_\phi = r\sin\theta\nabla\phi \tag{64}$$

$$\begin{array}{lll} \frac{\partial \boldsymbol{e}_r}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, & \frac{\partial \boldsymbol{e}_r}{\partial \phi} = \sin\theta\boldsymbol{e}_\phi, \\ \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, & \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = \cos\theta\boldsymbol{e}_\phi, \\ \frac{\partial \boldsymbol{e}_\phi}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_\phi}{\partial \theta} = 0, & \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta\boldsymbol{e}_r - \cos\theta\boldsymbol{e}_\theta \end{array} \tag{65}$$

$$\begin{array}{lcllcllcl} \nabla\boldsymbol{e}_r &=& \frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{1}{r}\boldsymbol{e}_\phi\boldsymbol{e}_\phi, & \nabla\cdot\boldsymbol{e}_r &=& \frac{1}{r}, & \nabla\times\boldsymbol{e}_r &=& 0, \\ \nabla\boldsymbol{e}_\theta &=& -\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi, & \nabla\cdot\boldsymbol{e}_\theta &=& \frac{1}{r}\frac{\cos\theta}{\sin\theta}, & \nabla\times\boldsymbol{e}_\theta &=& \frac{1}{r}\boldsymbol{e}_\phi, \\ \nabla\boldsymbol{e}_\phi &=& -\frac{1}{r}\boldsymbol{e}_\phi\boldsymbol{e}_r - \frac{1}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\theta, & \nabla\cdot\boldsymbol{e}_\phi &=& 0, & \nabla\times\boldsymbol{e}_\phi &=& \frac{\cos\theta}{r\sin\theta}\boldsymbol{e}_r - \frac{1}{r}\boldsymbol{e}_\theta \end{array} \tag{66}$$

勾配

$$\nabla f = \boldsymbol{e}_r \frac{\partial f}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \tag{67}$$

$$\begin{aligned}
\nabla \boldsymbol{g} \quad = \quad & \left(\boldsymbol{e}_r \frac{\partial g_r}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_r}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial g_r}{\partial \phi}\right)\boldsymbol{e}_r \\
& + \left(\boldsymbol{e}_r \frac{\partial g_\theta}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial g_\theta}{\partial \phi}\right)\boldsymbol{e}_\theta \\
& + \left(\boldsymbol{e}_r \frac{\partial g_\phi}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_\phi}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial g_\phi}{\partial \phi}\right)\boldsymbol{e}_\phi \\
& + g_r\left(\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{1}{r}\boldsymbol{e}_\phi\boldsymbol{e}_\phi\right) + g_\theta\left(-\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi\right) + g_\phi\left(-\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r - \frac{1}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\theta\right) \\
= \quad & \begin{pmatrix}
\frac{\partial g_r}{\partial r} & \frac{\partial g_\theta}{\partial r} & \frac{\partial g_\phi}{\partial r} \\
\frac{1}{r}\frac{\partial g_r}{\partial \theta} - \frac{g_\theta}{r} & \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{g_r}{r} & \frac{1}{r}\frac{\partial g_\phi}{\partial \theta} \\
\frac{1}{r\sin\theta}\frac{\partial g_r}{\partial \phi} - \frac{g_\phi}{r} & \frac{1}{r\sin\theta}\frac{\partial g_\theta}{\partial \phi} - \frac{g_\phi\cos\theta}{r\sin\theta} & \frac{1}{r\sin\theta}\frac{\partial g_\phi}{\partial \phi} + \frac{g_r}{r} + \frac{g_\theta\cos\theta}{r\sin\theta}
\end{pmatrix}
\end{aligned} \tag{68}$$

発散、回転

$$\nabla \cdot \boldsymbol{g} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(g_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial g_\phi}{\partial \phi} \tag{69}$$

$$\begin{aligned}
\nabla \times \boldsymbol{g} \quad = \quad & \boldsymbol{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(g_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial g_\theta}{\partial \phi}\right) + \\
& \boldsymbol{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial g_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r g_\phi)\right) + \\
& \boldsymbol{e}_\phi\left(\frac{1}{r}\frac{\partial}{\partial r}(r g_\theta) - \frac{1}{r}\frac{\partial g_r}{\partial \theta}\right)
\end{aligned} \tag{70}$$

ラプラシアン

$$\Delta r = \frac{2}{r}, \quad \Delta\theta = \frac{\cos\theta}{r^2\sin\theta}, \quad \Delta\phi = 0 \tag{71}$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial\phi} \tag{72}$$

$$
\begin{aligned}
\nabla \cdot (\nabla \boldsymbol{e}_r \cdot \boldsymbol{g}) &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(g_\theta \sin\theta) + \frac{\partial g_\phi}{\partial \phi}\right] && (73)\\
\nabla \cdot (\nabla \boldsymbol{e}_\theta \cdot \boldsymbol{g}) &= \frac{1}{r^2 \sin\theta}\left[-\frac{\partial}{\partial\theta}(g_r \sin\theta) + \frac{\cos\theta}{\sin\theta}\frac{\partial g_\phi}{\partial \phi}\right] && (74)\\
\nabla \cdot (\nabla \boldsymbol{e}_\phi \cdot \boldsymbol{g}) &= -\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\phi}(g_r \sin\theta + g_\theta \cos\theta) && (75)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}({}^t(\nabla \boldsymbol{g}) \cdot \nabla \boldsymbol{e}_r) &= \frac{1}{r^2 \sin\theta}\left[2g_r \sin\theta + \frac{\partial}{\partial\theta}(g_\theta \sin\theta) + \frac{\partial g_\phi}{\partial\phi}\right] && (76)\\
\mathrm{Tr}({}^t(\nabla \boldsymbol{g}) \cdot \nabla \boldsymbol{e}_\theta) &= \frac{1}{r^2}\left[-\frac{\partial g_r}{\partial\theta} + \frac{\cos\theta}{\sin\theta}g_r + \frac{1}{\sin^2\theta}g_\theta + \frac{\cos\theta}{\sin\theta^2}\frac{\partial g_\phi}{\partial\phi}\right] && (77)\\
\mathrm{Tr}({}^t(\nabla \boldsymbol{g}) \cdot \nabla \boldsymbol{e}_\phi) &= -\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial g_r}{\partial\phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial g_\theta}{\partial\phi} - \frac{1}{\sin^2\theta}g_\phi\right] && (78)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta \boldsymbol{g})_r &= \Delta g_r - \frac{2g_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(g_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial g_\phi}{\partial\phi} && (79)\\
(\Delta \boldsymbol{g})_\theta &= \Delta g_\theta + \frac{2}{r^2}\frac{\partial g_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial g_\phi}{\partial\phi} - \frac{g_\theta}{r^2\sin^2\theta} && (80)\\
(\Delta \boldsymbol{g})_\phi &= \Delta g_\phi + \frac{1}{r^2\sin^2\theta}\left[2\sin\theta\frac{\partial g_r}{\partial\phi} + 2\cos\theta\frac{\partial g_\theta}{\partial\phi} - g_\phi\right] && (81)
\end{aligned}
$$

4 準トロイダル座標

デカルト座標系 $\boldsymbol{x} = (x, y, z)$ から準トロイダル座標系 $\boldsymbol{x}' = (r, \theta, \phi)$ への変換 $\mathcal{F}_\mathrm{t} : \boldsymbol{x} \to \boldsymbol{x}'$ は

$$
\left\{
\begin{aligned}
x &= (R + r\cos\theta)\sin\phi\\
y &= (R + r\cos\theta)\cos\phi\\
z &= r\sin\theta
\end{aligned}
\right. \tag{82}
$$

で定義される。ここで、R, r はそれぞれ大半径、小半径であり、θ はポロイダル角、ϕ はトロイダル角である。

ヤコビアン

$$
J_{\mathcal{F}_\mathrm{t}} = \begin{pmatrix} \cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ -\frac{1}{r}\sin\theta\sin\phi & -\frac{1}{r}\sin\theta\cos\phi & -\frac{1}{r}\cos\theta \\ \frac{\cos\phi}{R+r\cos\theta} & -\frac{\sin\phi}{R+r\cos\theta} & 0 \end{pmatrix}, \quad |J_{\mathcal{F}_\mathrm{t}}| = \frac{1}{r(R+r\cos\theta)} \tag{83}
$$

$$
J_{\mathcal{F}_\mathrm{t}^{-1}} = \begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & (R+r\cos\theta)\cos\phi \\ \cos\theta\cos\phi & -r\sin\theta\cos\phi & -(R+r\cos\theta)\sin\phi \\ \sin\theta & r\cos\theta & 0 \end{pmatrix}, \quad |J_{\mathcal{F}_\mathrm{t}^{-1}}| = r(R+r\cos\theta) \tag{84}
$$

基本ベクトル

$$\nabla r = \begin{pmatrix} \cos\theta\sin\phi \\ \cos\theta\cos\phi \\ \sin\theta \end{pmatrix}, \quad \nabla\theta = \begin{pmatrix} -\frac{1}{r}\sin\theta\sin\phi \\ -\frac{1}{r}\sin\theta\cos\phi \\ \frac{1}{r}\cos\theta \end{pmatrix}, \quad \nabla\phi = \begin{pmatrix} \frac{\cos\phi}{R+r\cos\theta} \\ -\frac{\sin\phi}{R+r\cos\theta} \\ 0 \end{pmatrix} \tag{85}$$

$$\boldsymbol{e}_r = \nabla r, \quad \boldsymbol{e}_\theta = r\nabla\theta, \quad \boldsymbol{e}_\phi = (R + r\cos\theta)\nabla\phi \tag{86}$$

$$\begin{array}{lll} \frac{\partial \boldsymbol{e}_r}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, & \frac{\partial \boldsymbol{e}_r}{\partial \phi} = \cos\theta \boldsymbol{e}_\phi, \\ \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, & \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = -\sin\theta \boldsymbol{e}_\phi, \\ \frac{\partial \boldsymbol{e}_\phi}{\partial r} = 0, & \frac{\partial \boldsymbol{e}_\phi}{\partial \theta} = 0, & \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\cos\theta \boldsymbol{e}_r + \sin\theta \boldsymbol{e}_\theta \end{array} \tag{87}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\nabla x, \quad x = r\cos\theta \tag{88}$$

$$\nabla \boldsymbol{e}_r = \frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\cos\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi, \tag{89}$$

$$\nabla \boldsymbol{e}_\theta = -\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r - \frac{\sin\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi, \tag{90}$$

$$\nabla \boldsymbol{e}_\phi = -\frac{\cos\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_r + \frac{\sin\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\theta \tag{91}$$

$$\begin{array}{rclrcl} \nabla\cdot\boldsymbol{e}_r &=& \frac{1}{r} + \frac{\cos\theta}{R+r\cos\theta}, & \nabla\times\boldsymbol{e}_r &=& 0, \\ \nabla\cdot\boldsymbol{e}_\theta &=& -\frac{\sin\theta}{R+r\cos\theta}, & \nabla\times\boldsymbol{e}_\theta &=& \frac{1}{r}\boldsymbol{e}_\phi, \\ \nabla\cdot\boldsymbol{e}_\phi &=& 0, & \nabla\times\boldsymbol{e}_\phi &=& -\frac{\sin\theta}{R+r\cos\theta}\boldsymbol{e}_r - \frac{\cos\theta}{R+r\cos\theta}\boldsymbol{e}_\theta \end{array} \tag{92}$$

$$\nabla\times\boldsymbol{e}_\phi = \frac{1}{R+x}\nabla x \times \boldsymbol{e}_\phi \tag{93}$$

勾配

$$\nabla f = \boldsymbol{e}_r\frac{\partial f}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \boldsymbol{e}_\phi\frac{1}{R + r\cos\theta}\frac{\partial f}{\partial \phi} \tag{94}$$

$$
\begin{aligned}
\nabla \boldsymbol{g} &= \left(\boldsymbol{e}_r \frac{\partial g_r}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_r}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{R + r\cos\theta}\frac{\partial g_r}{\partial \phi}\right)\boldsymbol{e}_r \\
&\quad + \left(\boldsymbol{e}_r \frac{\partial g_\theta}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{R + r\cos\theta}\frac{\partial g_\theta}{\partial \phi}\right)\boldsymbol{e}_\theta \\
&\quad + \left(\boldsymbol{e}_r \frac{\partial g_\phi}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial g_\phi}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{R + r\cos\theta}\frac{\partial g_\phi}{\partial \phi}\right)\boldsymbol{e}_\phi \\
&\quad + g_r \left(\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\cos\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi\right) + g_\theta \left(-\frac{1}{r}\boldsymbol{e}_\theta\boldsymbol{e}_r - \frac{\sin\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\phi\right) \\
&\quad + g_\phi \left(-\frac{\cos\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_r + \frac{\sin\theta}{R + r\cos\theta}\boldsymbol{e}_\phi\boldsymbol{e}_\theta\right) \\
&= \begin{pmatrix}
\frac{\partial g_r}{\partial r} & \frac{\partial g_\theta}{\partial r} & \frac{\partial g_\phi}{\partial r} \\
\frac{1}{r}\frac{\partial g_r}{\partial \theta} - \frac{g_\theta}{r} & \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{g_r}{r} & \frac{1}{r}\frac{\partial g_\phi}{\partial \theta} \\
\frac{1}{R+r\cos\theta}\left(\frac{\partial g_r}{\partial \phi} - g_\phi\cos\theta\right) & \frac{1}{R+r\cos\theta}\left(\frac{\partial g_\theta}{\partial \phi} + g_\phi\sin\theta\right) & \frac{1}{R+r\cos\theta}\left(\frac{\partial g_\phi}{\partial \phi} + g_r\cos\theta - g_\theta\sin\theta\right)
\end{pmatrix}
\end{aligned}
\tag{95}
$$

発散、回転

$$
\nabla \cdot \boldsymbol{g} = \frac{1}{(R + r\cos\theta)}\left[\frac{1}{r}\frac{\partial}{\partial r}(r(R + r\cos\theta)g_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(R + r\cos\theta)g_\theta + \frac{\partial g_\phi}{\partial \phi}\right] \tag{96}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{g} &= \boldsymbol{e}_r \left(\frac{1}{r}\frac{\partial g_\phi}{\partial \theta} - \frac{1}{R + r\cos\theta}\left(\frac{\partial g_\theta}{\partial \phi} + g_\phi \sin\theta\right)\right) + \\
&\quad \boldsymbol{e}_\theta \left(\frac{1}{R + r\cos\theta}\left(\frac{\partial g_r}{\partial \phi} - g_\phi\cos\theta\right) - \frac{\partial g_\phi}{\partial r}\right) + \\
&\quad \boldsymbol{e}_\phi \left(\frac{1}{r}\frac{\partial}{\partial r}(r g_\theta) - \frac{1}{r}\frac{\partial g_r}{\partial \theta}\right)
\end{aligned}
\tag{97}
$$

ラプラシアン

$$
\Delta r = \frac{1}{r} + \frac{\cos\theta}{R + r\cos\theta}, \quad \Delta\theta = -\frac{\sin\theta}{r(R + r\cos\theta)}, \quad \Delta\phi = 0 \tag{98}
$$

$$
\begin{aligned}
\Delta f &= \frac{1}{r(R + r\cos\theta)}\frac{\partial}{\partial r}\left(r(R + r\cos\theta)\frac{\partial f}{\partial r})\right) + \frac{1}{r^2(R + r\cos\theta)}\frac{\partial}{\partial \theta}\left((R + r\cos\theta)\frac{\partial f}{\partial \theta}\right) \\
&\quad + \frac{1}{(R + r\cos\theta)^2}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}
\tag{99}
$$

$$\begin{aligned}
\nabla\cdot(\nabla\boldsymbol{e}_r\cdot\boldsymbol{g}) &= \frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{g_\theta\sin\theta}{r(R+r\cos\theta)}+\frac{\cos\theta}{(R+r\cos\theta)^2}\frac{\partial g_\phi}{\partial\phi} && (100)\\
\nabla\cdot(\nabla\boldsymbol{e}_\theta\cdot\boldsymbol{g}) &= -\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}+\frac{g_r\sin\theta}{r(R+r\cos\theta)}-\frac{\sin\theta}{(R+r\cos\theta)^2}\frac{\partial g_\phi}{\partial\phi} && (101)\\
\nabla\cdot(\nabla\boldsymbol{e}_\phi\cdot\boldsymbol{g}) &= \frac{1}{(R+r\cos\theta)^2}\left[-\cos\theta\frac{\partial g_r}{\partial\phi}+\sin\theta\frac{\partial g_\theta}{\partial\phi}\right] && (102)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_r) &= \frac{g_r}{r^2}+\frac{1}{r^2}\frac{\partial g_\theta}{\partial\theta}+\frac{\cos\theta}{(R+r\cos\theta)^2}\left[\frac{\partial g_\phi}{\partial\phi}+g_r\cos\theta-g_\theta\sin\theta\right] && (103)\\
\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_\theta) &= \frac{g_\theta}{r^2}-\frac{1}{r^2}\frac{\partial g_r}{\partial\theta}-\frac{\sin\theta}{(R+r\cos\theta)^2}\left[\frac{\partial g_\phi}{\partial\phi}+g_r\cos\theta-g_\theta\sin\theta\right] && (104)\\
\mathrm{Tr}({}^t(\nabla\boldsymbol{g})\cdot\nabla\boldsymbol{e}_\phi) &= \frac{1}{(R+r\cos\theta)^2}\left[-\cos\theta\frac{\partial g_r}{\partial\phi}+\sin\theta\frac{\partial g_\theta}{\partial\phi}+g_\phi\right] && (105)
\end{aligned}$$

$$\begin{aligned}
(\Delta\boldsymbol{g})_r &= \Delta g_r-\frac{g_r}{r^2}-\frac{2}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{2\cos\theta}{(R+r\cos\theta)^2}\frac{\partial g_\phi}{\partial\phi}\\
&\quad -\frac{g_r\cos^2\theta}{(R+r\cos\theta)^2}+\frac{g_\theta\cos\theta\sin\theta}{(R+r\cos\theta)^2}+\frac{g_\theta\sin\theta}{r(R+r\cos\theta)} && (106)\\
(\Delta\boldsymbol{g})_\theta &= \Delta g_\theta-\frac{g_\theta}{r^2}+\frac{2}{r^2}\frac{\partial g_r}{\partial\theta}+\frac{2\sin\theta}{(R+r\cos\theta)^2}\frac{\partial g_\phi}{\partial\phi}\\
&\quad +\frac{g_r\sin\theta\cos\theta}{(R+r\cos\theta)^2}-\frac{g_r\sin\theta}{r(R+r\cos\theta)}-\frac{g_\theta\sin^2\theta}{(R+r\cos\theta)^2} && (107)\\
(\Delta\boldsymbol{g})_\phi &= \Delta g_\phi-\frac{1}{(R+r\cos\theta)^2}\left[g_\phi-2\cos\theta\frac{\partial g_r}{\partial\phi}+2\sin\theta\frac{\partial g_\theta}{\partial\phi}\right] && (108)
\end{aligned}$$

参考文献

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