and

$$< e^{ik_1 X(z_1)} \cdots\cdots e^{ik_n X(z_n)} >$$

$$= \prod_{i<j} (z_i - z_j)^{k_i k_j} \quad \delta(k_1 + \cdots + k_n)$$

(where we used $<k|k'> = \delta(k-k')$

Eg

$$< e^{ik_1 X(z_1)} \; e^{ik_2 X(z_2)} > =$$

$$= \delta(k_1+k_2) \; (z_1 - z_2)^{k_1 k_2} = \delta(k_1+k_2) \; \frac{1}{(z_1-z_2)^{2h_1}}$$

$$< e^{ik_1 X(z_1)} \cdots e^{ik_3 X(z_3)} > = z_{12}^{k_1 k_2} \; z_{13}^{k_1 k_3} \; z_{23}^{k_2 k_3} \; \delta(\Sigma k_i)$$

and note that $k_1 k_2 = \frac{1}{2}(k_1 + k_2)^2 - \frac{1}{2}(k_1^2 + k_2^2) = h_3 - h_1 - h_2$ etc

OTHER CFT's

(i) D FREE SCALAR FIELDS : $c = D$, $:e^{ik.X(z)}:$ conformal dimension $\frac{1}{2} k.k = \frac{1}{2} k^2$ etc.

(ii) FREE FIELD + BACKGROUND CHARGE

$$T(z) = -\frac{1}{2} :\partial X(z) \partial X(z): + iQ \, \partial^2 X(z)$$

Exercise $c = 1 - 12 Q^2$

$\left(\text{or } [\gamma_m, \beta_n] = \delta_{m+n,0} \right)$

(iii) $\beta\gamma$ system : $R(\gamma(z)\,\beta(w)) = \frac{1}{z-w}$, $R(\beta(z)\,\gamma(w)) = -\frac{1}{z-w}$

$$T(z) = (1-\lambda)\,\gamma\,\partial\beta - \lambda\,\partial\gamma\,\beta = \gamma\partial\beta - \lambda\,\partial(\gamma\beta)$$

then

$$T(z)\,\beta(w) = \frac{(1-\lambda)\,\partial\beta(z)}{z-w} + \frac{\lambda\,\beta(z)}{(z-w)^2} + \cdots$$

$$= \frac{\lambda\,\beta(w)}{(z-w)^2} + \frac{\partial\beta(w)}{z-w} + \cdots$$

and similarly

$$T(z)\,\gamma(w) = \frac{(1-\lambda)\,\gamma(z)}{(z-w)^2} + \frac{\lambda\,\partial\gamma(z)}{z-w} + \cdots$$

$$= \frac{(1-\lambda)\,\gamma(w)}{(z-w)^2} + \frac{\partial\gamma(w)}{z-w} + \cdots$$

$\Rightarrow$ (β,γ) pair of conformal dimension $(\lambda, 1-\lambda)$

Claim: $c = 2\left((1-\lambda)^2 - 4\lambda(1-\lambda) + \lambda^2\right)$

$= 2\left(1 - 6\lambda(1-\lambda)\right)$ (note $c = 26$ for $\lambda = 2$)

VIRASORO REPRESENTATION THEORY

$$\begin{cases} [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\ [c, L_n] = 0 \end{cases}$$

We have an anti-linear, anti-involution $\omega(L_n) = L_{-n}$
$\omega(c) = c$

(physics notation $L_n^+ = \omega(L_n)$)

(ie. $\omega(\lambda x) = \bar{\lambda}\omega(x)$ and $[\omega(x), \omega(y)] = -\omega([x,y])$)

Defn: A positive energy representation of Vir is a representation such that L_0 is diagonalizable and its eigenvalues are non negative

Defn: let V be a representation of Vir, $\langle\, ,\rangle$ a Hermitean form on V
$\langle\, ,\rangle$ is contravariant if $\langle xu, v\rangle = \langle u, \omega(x) v\rangle$ $\forall u,v \in V$, $x \in Vir$
V is unitary if $\langle v, v\rangle > 0$ $\forall v \in V \setminus \{0\}$

Note that if $L_0|\psi\rangle = h|\psi\rangle$ then

$$L_0 L_n |\psi\rangle = (L_n L_0 - n L_n)|\psi\rangle = (h-n) L_n|\psi\rangle$$

so L_0-eigenvalue of $L_n|\psi\rangle$ is n units lower. Hence a positive energy representation has (at least) one vector $|\phi\rangle$ st $L_n|\phi\rangle = 0$ $\forall n > 0$.

This leads us to:

Defn A highest weight representation (module) is a representation V st there exists a vector $|h\rangle \in V$ satisfying

$\hat{c}|h\rangle = c|h\rangle$

for some $c, h \in \mathbb{C}$.

$L_n|h\rangle = 0 \quad n>0$

$|h\rangle$ is called highest weight vector.

$L_0|h\rangle = h|h\rangle$

and $V = \mathcal{U}(Vir_-)|h\rangle = \text{span}\{L_{-i_k} \cdots L_{-i_1}|h\rangle\}$

When are hwm's unitary? (remark: unitary hwm's are irreducible)

Consider e.g. $[L_n, L_{-n}] = 2n L_0 + \frac{c}{12} n(n^2-1)$

ie for $|\psi\rangle = L_{-n}|h\rangle$ we have $\langle\psi|\psi\rangle =$

$\langle h|L_{-n}^+ L_{-n}|h\rangle = \left(2nh + \frac{c}{12} n(n^2-1)\right)\langle h|h\rangle$

taking $n=1$ gives $h \geq 0$

then n large gives $c \geq 0$.

__Defn__ The character of a hwm V is defined as

$$\text{ch}\, V = \dim_q V = \text{Tr}_V\, q^{L_0} = \sum_{j\geq 0} (\dim V_{h+j})\, q^{h+j}$$

where $V_{h+j} = \text{span}\{L_{-i_1} \cdots L_{-i_k}|h\rangle \quad i_1 + \cdots + i_k = j\}$

$= \{|\psi\rangle \in V \mid L_0|\psi\rangle = (h+j)|\psi\rangle\}$

Verma modules $M(c,h)$ and irreducible hwm's $V(c,h)$

Defn : If all vectors $L_{-i_k} \cdots L_{-i_1} |h\rangle$ are linearly independent,

then the hwm is called a Verma module $M(c,h)$

If V is a hwm, Then we have a homomorphism

$$M(c,h) \twoheadrightarrow V$$

$$|h\rangle_M \longrightarrow |h\rangle_V$$

ie. V is isomorphic to a quotient of $M(c,h)$

Note that $M(c,h) = \mathcal{U}(Vir) / I(c,h)$

where $I(c,h) = \text{span}\ \{ L_{n},\ n>0 ,\ L_0 - h\mathbb{1} ,\ c - c\mathbb{1} \}$

or $M(c,h) = \mathcal{U}(Vir) \otimes_{Vir_{\geq 0}} |h\rangle$

Proposition $M(c,h) = \bigoplus_{j\geq 0} M(c,h)_{j+h}$ (L_0 eigenspace decomp.)

where $M(c,h)_{h+j} = \text{span} \{ L_{-i_k} \cdots L_{-i_1} |h\rangle ,\ 0 < i_1 \leq \cdots \leq i_k ,\ i_1 + \cdots + i_k = j \}$

(a) $\text{ch}\ M(c,h) \equiv \text{Tr}_M\, q^{L_0} = \sum_{j\geq 0} \dim M(c,h)_{j+h}\ q^{j+h}$

$$= \frac{q^h}{\varphi(q)} \quad \text{with} \quad \varphi(q) = \prod_{k\geq 1} (1-q^k)$$

(b) $M(c,h)$ is indecomposable $(\neq V \oplus W)$

(c) $M(c,h)$ has a unique maximal proper submodule $J(c,h)$ such that

$V(c,h) = M(c,h)/J(c,h)$ is the unique irreducible hwm with

highest weight (c,h)

Pf (a) Clearly $\operatorname{ch} M(c,h) = q^h \sum_{j\geq 0} p(j)\, q^j$ where $p(j)$ is

the number of partitions of j.

$$\sum_{j\geq 0} p(j)\, q^j = \sum_{j\geq 0} \sum_{j_1+2j_2+\dots=j} q^{j_1+2j_2+\dots} =$$

$$= \prod_{k\geq 1} \sum_{j_k=0}^{\infty} q^{kj_k} = \frac{1}{\prod_{k\geq 1}(1-q^k)} = \frac{1}{\varphi(q)}$$

(Remark $\eta(q) = q^{\frac{1}{24}}\varphi(q)$ is Eulers η-function. In terms of

$q = e^{2\pi i\tau}$ we have

$$\begin{cases} \eta(\tau+1) = e^{i\pi/12}\,\eta(\tau) \\ \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau) \end{cases}$$

$\sim$ these generate $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PSL(2,\mathbb{Z})$)

(b) $|h\rangle$ is either in V or W, since both are graded

(For any submodule $U \subset V$ we have $U = \bigoplus U_j = \bigoplus (U \cap V)_j$)

thus either $|h\rangle \in V$ or W, but then $M(c,h) = U(Vir_-)\,|h\rangle \subset V$ or W

(c) $J(c,h)$ is the sum of all proper submodules (ie submodules that do not contain $|h\rangle$)

Proposition

(a) $M(c,h)$ carries a unique contravariant Hermitean form $\langle\,,\rangle$ such that $\langle h|h\rangle = 1$

(b) $M_{h+k} \perp M_{h+\ell}$ for $k \neq \ell$

(c) $\operatorname{Ker}\langle\,,\rangle = J(c,h)$

(Hence $V(c,h) = M(c,h)/J(c,h)$ carries a unique contravariant Hermitean form s.t. $\langle h|h\rangle = 1$ which is nondegenerate)

Pf: (a)+(b)

$$\langle \overbrace{L_{-i_m} \cdots L_{-i_1}}^{P} |h\rangle \mid \overbrace{L_{-j_n} \cdots L_{-j_1}}^{Q} |h\rangle \rangle =$$

$$= \langle h| \underbrace{\omega(P)\, Q}_{\in \, Vir_- \otimes Vir_0 \otimes Vir_+ \ (PBW)} |h\rangle = \underbrace{\langle \omega(P)\, Q\rangle_{CT}}_{\text{constant term}}$$

Clearly contravariant (by construction), also Hermitean since

for $P = L_{-j_1} \cdots L_{-j_s} (L_0^k c^r) L_{i_1} \cdots L_{i_t}$

$\omega(P) = L_{-i_t} \cdots L_{-i_1} (L_0^k c^r) L_{j_s} \cdots L_{j_1}$

ie has same constant term.

Also clearly nonvanishing iff $j_1 + \dots + j_n = i_1 + \dots + i_m$

(c) $\operatorname{Ker}\langle\,,\rangle$ is a proper submodule. Moreover, if V a proper

$\{u \in M(c,h) \mid \langle u, w\rangle = 0 \ \ \forall w \in M\}$

then $V \subset \operatorname{Ker} \langle , \rangle$ since if $P|h\rangle \in V$ and

$Q|h\rangle \in M(c,h)$ then $\omega(Q)P|h\rangle \in V$ and if

$\langle Q|h\rangle, P|h\rangle\rangle = \langle \omega(Q)P\rangle_{CT} \neq 0$ then $|h\rangle \in V$ and

hence $V = M(c,h)$ □

Defn. A vector $|v\rangle \in V$ is called <u>singular</u> if it is nonzero and

$L_n|v\rangle = 0 \quad \forall\, n > 0$

Prop. A hwm V is irreducible iff there exist no other singular vectors than multiples of the hwv.

Defn. $M(c,h) = \bigoplus_{n \geq 0} M(c,h)_{h+n}$ then

$\det_n(c,h) = \det \langle , \rangle |_{M(c,h)_{h+n}}$ is called the Kac determinant.

(note $\dim M(c,h)_{h+n} = p(n)$)

Prop (a) $M(c,h)$ reducible (degenerate) iff $\det_n(c,h) = 0$ for some n.

(b) $V(c,h)$ is unitary iff $\det_n(c,h) \geq 0 \quad \forall\, n \in \mathbb{Z}_{\geq 0}$

Pf: exercise.

$$\det_0(c,h) = \langle h | h \rangle = 1$$

$$\det_1(c,h) = \langle L_{-1} v_0 , L_{-1} v_0 \rangle = \langle h | L_1 L_{-1} | h \rangle = 2h \qquad v_0 = |h\rangle$$

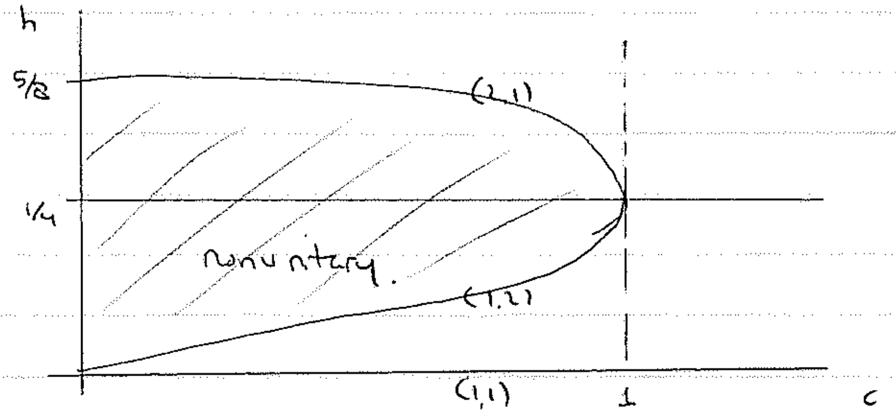
$$\det_2(c,h) = \begin{vmatrix} \langle h | L_2 L_{-2} | h \rangle & \langle h | L_2 L_{-1} L_{-1} | h \rangle \\ \langle h | L_1 L_1 L_{-2} | h \rangle & \langle h | L_1 L_1 L_{-1} L_{-1} | h \rangle \end{vmatrix}$$

$$= \begin{vmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{vmatrix}$$

$$= 2h\,(16h^2 + 2hc - 10h + c)$$

Note $16h^2 + 2hc - 10h + c = (4h-1)^2 + (2h+1)(c-1)$

$$\Rightarrow \quad 0 \leq c < 1 - \frac{(4h-1)^2}{2h+1} \qquad h \geq 0$$



Note $\det_2(c,h) = 32\,(h - h_{1,1})(h - h_{1,2})(h - h_{2,1})$

where $$h_{r,s}(c) = \frac{1}{48}\Big((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}\,(r^2-s^2) - 24rs - 2(1-c) \Big)$$

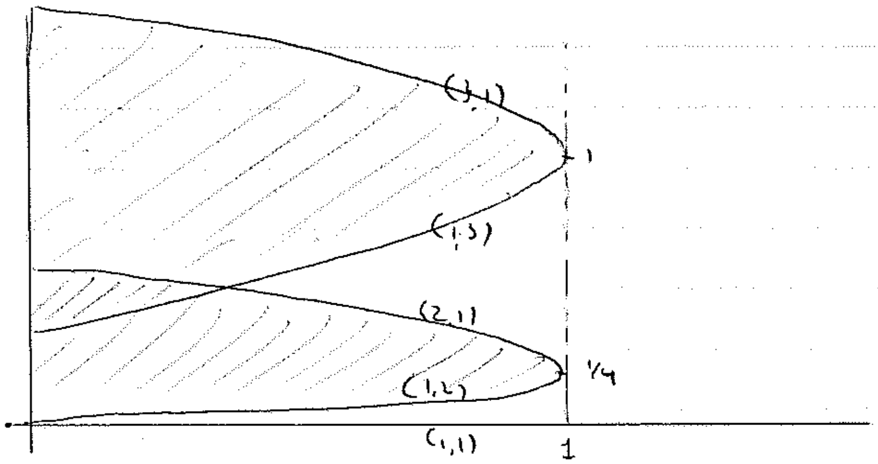
$$= \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

where $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$ or $c = 1 - \frac{6}{m(m+1)}$

Theorem

$$\det_n(c,h) = K \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq n}} \left(h - h_{r,s}(c)\right)^{p(n-rs)}$$

Eg.



Corollary

(a) $V(c,h)$ is unitary for $c \geq 1$, $h \geq 0$

(b) $V(c,h) \simeq M(c,h)$ for $c > 1$, $h > 0$

(c) $V(1,h) \simeq M(1,h)$ iff $h \neq \frac{m^2}{4}$ $m \in \mathbb{Z}$

Pf: For $c = 1$ $\det_n(1,h) = K \prod_{\substack{r,s \in \mathbb{N} \\ 1 \leq rs \leq n}} \left(h - \frac{(r-s)^2}{4}\right)^{p(n-rs)}$

(d) $V(c,h)$ is unitary in the range $c<1$ iff

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots$$

$$h = h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \qquad r = 1, \ldots, m-1, \quad s = 1, \ldots, m$$

$r \backslash s$	1	2	3
1	0	$\frac{1}{16}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{16}$	0

$m = 3$, $c = \frac{1}{2}$

(Ising model)

(Note $h_{r,s} = h_{m-r,(m+1)-s}$)