CORRELATION FUNCTIONS

$$\langle A_1(z_1,\bar z_1)\ldots A_n(z_n,\bar z_n)\rangle = \langle 0|\, R(A_1(z_1,\bar z_1)\ldots A_n(z_n,\bar z_n))|0\rangle$$

where $|0\rangle$ is the (unique) vacuum state $L_n|0\rangle = 0 \quad n\geq -1$

(Also $L_n^\dagger = L_{-n}$)

If $[L_n, \phi(z)] = (z^{n+1}\partial + h(n+1)z^n)\phi(z)$ for $n = 0, \pm 1$

then consider $\langle \phi_1(z)\, \phi_2(w)\rangle = G(z,w)$

L_{-1}: $0 = \langle 0| L_{-1}\phi_1(z)\phi_2(w)|0\rangle = (\partial_z + \partial_w)\, G(z,w)$

hence $G(z,w) = G(z-w)$

L_0: $0 = \left((z\partial_z + h_1) + (w\partial_w + h_2)\right) G(z,w)$

$$\Rightarrow (z\partial + (h_1+h_2))\, G(z) = 0 \Rightarrow G(z) \sim z^{-(h_1+h_2)}$$

L_1: $0 = \left((z^2\partial_z + 2h_1 z) + (w^2\partial_w + 2h_2 w)\right) G(z,w)$

$$\Rightarrow G(z) \sim \delta_{h_1 h_2}\, z^{-2h_1}$$

I.e. $\langle \phi_1(z,\bar z)\, \phi_2(w,\bar w)\rangle \approx (z-w)^{-2h_1}(\bar z-\bar w)^{-2h_2}\, \delta_{h_1 h_2}$

Further results $\langle \phi_1(z_1,\bar z_1)\ldots \phi_3(z_3,\bar z_3)\rangle \sim \prod_{i<j}(z_i - z_j)^{-h_{ij}^k} \times \text{c.c.}$

where $h_{ij}^k = h_i + h_j - h_k$

and $\langle \phi_1 \ldots \phi_4\rangle = (\text{prefactor}) \times f(x,\bar x)$, $x = \dfrac{z_{12}z_{34}}{z_{13}z_{24}}$ cross-ratio

$z_{ij} = z_i - z_j$

More precisely:

$$\langle \phi_1(z_1,\bar z_1) \dots \phi_4(z_4,\bar z_4) \rangle = \prod_{i<j} z_{ij}^{-h_i - h_j + h/3} \; \bar z_{ij}^{-\bar h_i - \bar h_j + \bar h/3}$$

$$\times \; f(x, \bar x)$$

$h = \sum h_i$, $\bar h = \sum \bar h_i$ $\qquad x = \dfrac{z_{12} z_{34}}{z_{13} z_{24}}$ $\qquad \bar x = \dfrac{\bar z_{12} \bar z_{34}}{\bar z_{13} \bar z_{24}}$ $\qquad$ cross ratio

Interpretation: Using the Möbius group one can put three points in arbitrary position, Thus correlation function only a function of $n-3$ independent coordinates, which are invariant under Möbius gp. These are the cross ratios.

Indeed $\quad z_{ij}' = z_i' - z_j' = \dfrac{a z_i + b}{c z_i + d} - \dfrac{a z_j + b}{c z_j + d} = \dfrac{z_{ij}}{(c z_i + d)(c z_j + d)}$

One often chooses $z_1 = \infty$, $z_2 = 1$, $z_4 = 0$ then $x = z_3$.

OPERATOR – STATE CORRESPONDENCE

$$|\phi\rangle = \lim_{\substack{z\to 0\\ \bar z \to 0}} \phi(z,\bar z)|0\rangle \quad \text{and v.v.} \quad \phi(z,\bar z)|0\rangle = e^{z L_{-1} + \bar z \bar L_{-1}} |\phi\rangle$$

For a primary field $\quad [L_n, \phi(z)] = (z^{n+1}\partial + h(n+1) z^n)\phi(z)$

hence $\quad L_n|\phi\rangle = 0 \quad n>0, \qquad L_0|\phi\rangle = h\,|\phi\rangle$

Note also that (exercise)

$$\langle\phi| = \lim_{z,\bar z \to \infty} z^{2h} \bar z^{2\bar h} \langle 0| \phi(z,\bar z)$$

With $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h}$ we find

$$\phi_n |0\rangle = 0 \qquad n > -h$$

$$\phi_{-h} |0\rangle = |\phi\rangle$$

in particular $|T\rangle = \lim_{z\to 0} T(z)|0\rangle = L_{-2}|0\rangle$

while $L_n|0\rangle = 0$ for $n \geq -1$.

FREE SCALAR FIELD

We have seen that

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \; \partial_\alpha X \,\partial^\alpha X \qquad \text{put } \alpha' = 2, \quad d^2\sigma = d\sigma\, d\tau$$

$$= \frac{1}{2\pi}\int d^2 z \;\; \partial X\,\bar\partial X$$

gives classical solution

$$X(z,\bar z) = X(z) + \bar X(\bar z)$$

$$X(z) = \tfrac{1}{2} q - i p \log z + i \sum_{n\neq 0} \tfrac{1}{n}\alpha_n z^{-n}$$

$$\bar X(\bar z) = \tfrac{1}{2} q - i p \log \bar z + i \sum_{n\neq 0} \tfrac{1}{n}\bar\alpha_n \bar z^{-n}$$

and, upon quantization, $[\alpha_m, \alpha_n] = m\,\delta_{m+n,0}$ $\quad (p = \alpha_0)$

$$[q, p] = i$$

$X(z)$ not a conformal field, but $i\partial X = \sum \alpha_n z^{-n-1}$ is.

Also, we have seen that

$$T_{zz}(z) = T(z) = \tfrac{1}{2} : i\partial X(z)\, i\partial X(z) : = \sum_n \underbrace{\Big(\tfrac{1}{2}\sum_m \alpha_{n-m}\alpha_m\Big)}_{L_n} z^{-n-2}$$

Now, observe (N.O. puts p to right of q)

$$R\big(X(z,\bar z) X(w,\bar w)\big) = \; : X(z,\bar z) X(w,\bar w) : \; - i[p,q]\,(\log z + \log \bar z)$$

$$+ \Big[i\sum_{n>0} \tfrac{1}{n}\alpha_n z^{-n} \,,\; i\sum_{m<0} \tfrac{1}{n}\alpha_m w^{-m} \Big] + (\text{anti-hol.})$$

commutator yields

$$-\sum_{\substack{n>0\\ m<0}} \frac{1}{nm}\,[\alpha_n,\alpha_m]\, z^{-n} w^{-m}$$

$$= \sum_{n>0} \frac{1}{n}\left(\frac{w}{z}\right)^n = -\log\left(1-\frac{w}{z}\right) \quad \text{for } |z|>|w|$$

then

$$R\left(X(z,\bar z)\, X(w,\bar w)\right) = :X(z,\bar z)\, X(w,\bar w): - \log(z-w) - \log(\bar z-\bar w)$$

and

$$R\left(i\partial X(z)\; i\partial X(w)\right) = \frac{1}{(z-w)^2} + \underbrace{:i\partial X(z)\; i\partial X(w):}_{\text{regular for } z\to w}$$

ie. $$\langle i\partial X(z)\; i\partial X(w)\rangle = \frac{1}{(z-w)^2} \equiv \underbrace{i\partial X(z)\;\; i\partial X(w)}$$

and

$$T(z) = \lim_{z\to w} \frac{1}{2}\left(R\left(i\partial X(z)\, i\partial X(w)\right) - \frac{1}{(z-w)^2}\right)$$

<u>WICK THEOREM</u> $\quad = :\ :$ (for $[A_>, B_<]$ c-numbers)

$$R(ABC\ldots) = N(ABC\ldots) + N(\underbrace{AB}C\ldots) + N(A\underbrace{BC}\ldots)$$

$$+ \ldots + N(\underbrace{AB}\,\underbrace{CD}\ldots) + \ldots$$

In $R(N(AB\ldots)\,N(\ldots)\ldots)$ omit contractions between operators which are already normal ordered.

Consequences,

$$R\left(T(z)\ i\partial X(w)\right) = R\left(\tfrac{1}{2} :i\partial X(z)\ i\partial X(z): \ i\partial X(w)\right)$$

$$= \frac{i\partial X(z)}{(z-w)^2} + \text{reg} = \frac{i\partial X(w)}{(z-w)^2} + \frac{i\partial^2 X(w)}{z-w} + \text{reg}$$

ie. $i\partial X(z)$ is primary field of conformal dimension 1.

$$R\left(T(z)T(w)\right) = \frac{1}{4} R\left(:i\partial X(z)\ i\partial X(z): \ :i\partial X(w)\ i\partial X(w):\right)$$

$$= \frac{1/2}{(z-w)^4} + \frac{:i\partial X(z)\ i\partial X(w):}{(z-w)^2} + \text{reg}$$

$$= \frac{1/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg}$$

ie. Virasoro OPE with $c=1$.

<u>Claim</u>: $:e^{ikX(z,\bar z)}: = e^{ikX_<(z,\bar z)}\, e^{ikq}\, z^{kp}\, \bar z^{kp}\, e^{ikX_>(z,\bar z)}$

is a primary field of conformal dimension $(h,\bar h) = \left(\frac{k^2}{2}, \frac{k^2}{2}\right)$

Proof: $R\left(T(z)\ :(ikX(w))^n:\right) = -\frac{1}{2} :\partial X(z)\partial X(z): \ :(ikX(w))^n:$

$= \left(\text{using } \underbrace{\partial X(z)\ X(w)} = \frac{-1}{(z-w)}\right)$

$$= \frac{\frac{k^2}{2} n(n-1) (ikX(w))^{n-2}}{(z-w)^2} + \frac{2\left(\frac{1}{2}ik\right) n\ (ikX(w))^{n-1}\, \partial X(z)}{z-w}$$

$+ \text{reg}$

$\Rightarrow$

$$R\left(T(z)\, :e^{ikX(w)}:\right) = \frac{\frac{k^2}{2} :e^{ikX(w)}:}{(z-w)^2} + \frac{ik :\partial X(z)\, e^{ikX(w)}:}{z-w} + \dots$$

$$= \frac{\frac{k^2}{2}\, e^{ikX(w)}}{(z-w)^2} + \frac{\partial e^{ikX(w)}}{z-w} + \dots$$

Note also that, using $e^A e^B = e^B e^A e^{[A,B]}$ if $[A,B]$ is c-number

$$:e^{ik_1X(z)}: :e^{ik_2X(w)}: = (z-w)^{k_1k_2} :e^{ik_1X(z)}\, e^{ik_2X(w)}:$$

(as we have seen in the computation of the Veneziano amplitude,

or by using $[X_{>}(z), X_{<}(w)] = -\log(z-w)$)

CORRELATION FUNCTIONS

$$\langle i\partial X(z_1)\, i\partial X(z_2)\rangle = \frac{1}{(z_1-z_2)^2}$$

$$\langle i\partial X(z_1)\, i\partial X(z_2)\, i\partial X(z_3)\rangle = 0$$

$$\langle i\partial X(z_1) \dots i\partial X(z_4)\rangle = \frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{24}^2} + \frac{1}{z_{14}^2 z_{23}^2}$$

$$= (z_{12}\, z_{13}\, z_{14}\, z_{23}\, z_{24}\, z_{34})^{-2/3} \times f(x)$$

// where $f(x) = \left(x^{2/3}(1-x)^{2/3} + x^{-4/3}(1-x)^{2/3} + x^{2/3}(1-x)^{-4/3}\right)$