We have seen that the conformal Killing vector equation

$$D_\mu k_\nu + D_\nu k_\mu = \omega g_{\mu\nu}$$

for $g_{\mu\nu} = \delta_{\mu\nu}$ in 2D, has an infinite number of solutions,

given by the CR equations $\partial_1 k_1 = \partial_2 k_2$, $\partial_1 k_2 = -\partial_2 k_1$

(ie holomorphic or antiholomorphic transformations)

In terms of $k = k_1 + i k_2$ $\quad k(z\bar{z}) = k(z) + \bar{k}(\bar{z})$

$z = x^1 + i x^2$, $\quad \partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ $\quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$

$x^1 = \frac{1}{2}(z + \bar{z})$ $\quad$, $x^2 = \frac{1}{2i}(z - \bar{z})$

Where the generators corresponding to $z \to z + \epsilon z^{n+1}$

are given by $L_n = -z^{n+1}\frac{\partial}{\partial z}$.

TENSORS IN COMPLEX COORDINATES

$$J_z = J_\mu \frac{\partial x^\mu}{\partial z} = J_1 \frac{\partial x^1}{\partial z} + J_2 \frac{\partial x^2}{\partial z} = \frac{1}{2}(J_1 - i J_2)$$

$$J_{\bar{z}} = \qquad\qquad = \frac{1}{2}(J_1 + i J_2)$$

(just like $\partial_z$ and $\partial_{\bar{z}}$)

While, e.g. for a 2-tensor

$$T_{zz} = \frac{1}{4}\left(T_{11} - T_{22} - i\,(T_{12} + T_{21})\right)$$

$$T_{\bar z\bar z} = \frac{1}{4}\left(T_{11} - T_{22} + i\,(T_{12} + T_{21})\right)$$

$$T_{z\bar z} = \frac{1}{4}\left(T_{11} + T_{22} + i\,(T_{12} - T_{21})\right)$$

$$T_{\bar z z} = \frac{1}{4}\left(T_{11} + T_{22} - i\,(T_{12} - T_{21})\right)$$

In particular $g_{\mu\nu} = \delta_{\mu\nu} \Rightarrow g_{zz} = g_{\bar z\bar z} = 0 \qquad g_{z\bar z} = g_{\bar z z} = \frac{1}{2}$.

CONSERVED CURRENTS

In flat space : symmetry $\Rightarrow$ conserved current $\partial_\mu J^\mu = 0$
(Noether's theorem)

$\Downarrow$

generator of symmetry $\Leftarrow$ conserved charge $Q = \int d^d\dot{x}\; J^0$

$[Q, \phi] = \delta\phi$ quantity

In curved space, however, we need $\quad Q = \int d^d\dot{x}\; \sqrt{g}\; J^0$

i.e. we need $\quad \frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\, J^\mu\right) = D_\mu J^\mu = 0$

to have a conserved charge

CONSERVED CURRENTS (CONT'D)

Under variation of the metric

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g}\, \underbrace{T^{\mu\nu}}_{\text{energy-momentum tensor}} \delta g_{\mu\nu} \qquad \text{ie} \quad T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

general coordinate invariance implies

$$\nabla_\mu T^{\mu\nu} = 0$$

If the theory is in addition Weyl invariant, ie invariant under $\delta g_{\mu\nu} = \Omega\, g_{\mu\nu}$, then we have

$$g_{\mu\nu} T^{\mu\nu} = T^\mu{}_\mu = 0$$

However in curved spacetime $\nabla_\mu T^{\mu\nu} = 0$ doesn't automatically lead to a conserved current / charge, eg

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g}\, T^{\mu\nu} \right) + \underbrace{\Gamma^\nu{}_{\sigma\mu} T^{\sigma\mu}}_{\text{work done by gravity}}$$

However conserved charges exist for symmetries $J^\mu[K] = T^{\mu\nu} K_\nu$

$$\nabla_\mu J^\mu = \underbrace{(\nabla_\mu T^{\mu\nu})}_{=0} K_\nu + \frac{1}{2} T^{\mu\nu} \underbrace{(\nabla_\mu K_\nu + \nabla_\nu K_\mu)}_{=0}$$

for Killing vector

or conformal Killing vector if $T^\mu{}_\mu = 0$

In 2D CFT: $T_{\mu\nu}$ is symmetric and traceless, i.e. $T_{12}=T_{21}$, $T_{11}+T_{22}=0$

hence independent components are $T_{zz}$, $T_{\bar z\bar z}$ $\quad (T_{z\bar z}=0)$

while $\partial^\mu T_{\mu\nu} = g^{\mu\rho}\partial_\rho T_{\mu\nu} = 0$ implies

$$\partial_z T_{\bar z\bar z} = 0 = \partial_{\bar z} T_{zz}$$

Denote $T = T_{zz}$, $\bar T = T_{\bar z\bar z}$.

Conserved current corresponding to conformal transformations given by

$$J_\mu[k] = T_{\mu\nu}k^\nu \qquad k^z = k_1 + ik_2 = k(z), \quad k^{\bar z} = k_1 - ik_2 = \bar k(\bar z)$$

$$J_z = T_{zz}k^z = T(z)k(z)$$

$$J_{\bar z} = T_{\bar z\bar z}k^{\bar z} = \bar T(\bar z)\bar k(\bar z)$$

(clearly conserved since $\partial_{\bar z}J_z = 0 = \partial_z J_{\bar z}$)

For $k(z) = z^{n+1}$ we get charge

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \left(\text{i.e. } T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\right)$$

TENSORS

$$T_{\underbrace{z\ldots z}_{p}\underbrace{\bar z\ldots\bar z}_{q}} \to T_{\underbrace{w\ldots w}_{p}\underbrace{\bar w\ldots\bar w}_{q}}\left(\frac{\partial w}{\partial z}\right)^p\left(\frac{\partial \bar w}{\partial \bar z}\right)^q \qquad \text{for } z\to w(z), \ \bar z\to\bar w(\bar z)$$

More generally, a field transforming under conformal transformations as

$$\phi(z,\bar z) \to \phi(w,\bar w)\left(\frac{\partial w}{\partial z}\right)^h\left(\frac{\partial \bar w}{\partial \bar z}\right)^{\bar h}$$

is called a conformal field (primary field) of weight $(h, \bar{h})$

Infinitesimally, under $z \to z + \epsilon(z)$

$$\delta_\epsilon \phi(z,\bar{z}) = \left(\epsilon(z)\partial_z + h\,\partial_z\epsilon(z)\right)\phi(z,\bar{z})$$

(+ similar for $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$)

Under scaling : $x^\mu \to \lambda x^\mu$ ie. $z \to \lambda z$, $\bar{z} \to \lambda \bar{z}$

$$\phi(z,\bar{z}) \to e^{\lambda\Delta}\,\phi(e^\lambda z, e^\lambda \bar{z})$$

$$\Delta = h + \bar{h} = \text{scaling dimension} \qquad (\text{cf. } D)$$

Under rotations : $z \to e^{i\theta} z$, $\bar{z} \to e^{-i\theta}\bar{z}$

$$\phi(z,\bar{z}) \to e^{i\theta s}\,\phi(e^{i\theta}z, e^{-i\theta}\bar{z})$$

$$s = h - \bar{h} = \text{spin} \qquad (\text{cf. } iM)$$

(since fields should be single valued under $2\pi$ rotation $s \in \mathbb{Z}$)

If $\bar{h} = 0$ then $\phi(z,\bar{z}) = \phi(z)$. We expand

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n\, z^{-n-h}$$

[Note, on cylinder $\phi(w) = \sum_{n\in\mathbb{Z}} \phi_n\, e^{-in\overbrace{(\sigma+i\tau)}^{w}}$

where $\phi_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, \phi^{cyl}(\sigma,\tau)e^{in\sigma} = (i^h)\oint \frac{dz}{2\pi i} z^{n+h-1}\phi^{plane}(z)$

$$\phi^{plane}(z) = \phi^{cyl}(w)\left(\frac{dw}{dz}\right)^h = \phi^{cyl}(w)\,(iz)^{-h} \quad ]$$

$z = e^{iw}$ $\quad dz = iz\,dw$

# QUANTUM CONFORMAL FIELD THEORY

## RADIAL ORDERING

In order to make sense of product of operators, we need to radially order them (time ordering on the cylinder)

$$R\left( A(z,\bar{z})\, B(w,\bar{w}) \right) = \begin{cases} A(z,\bar{z})\, B(w,\bar{w}) & |z| > |w| \\ B(w,\bar{w})\, A(z,\bar{z}) & |z| < |w| \end{cases}$$

and correlation functions are defined as $\langle A_1(z_1,\bar{z}_1) \ldots A_n(z_n,\bar{z}_n) \rangle$

$$\equiv \underset{\substack{\uparrow \\ t\to\infty \text{ "out state"}}}{\langle 0|}\, R\left( A_1(z_1,\bar{z}_1) \ldots A_n(z_n,\bar{z}_n) \right) \underset{\substack{\uparrow \\ t\to-\infty \text{ "in state"}}}{|0\rangle}$$

For a primary field of weight $h$ we have (suppress $\bar{z}$ dependence)

$$\delta_\epsilon \phi(z) = \left( h\, \partial\epsilon(z) + \epsilon(z)\, \partial \right) \phi(z)$$

and this should equal, after quantization, $[Q_\epsilon, \phi(z)]$

with $Q_\epsilon = \frac{1}{2\pi i} \oint dz\, \epsilon(z)\, T(z)$.

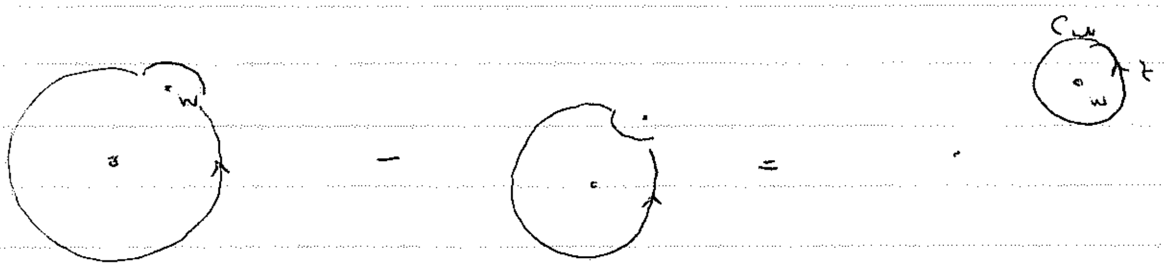
or, for $\epsilon(z) := z^{n+1}$

$$[L_n, \phi(z)] = \left( z^{n+1}\, \partial + h(n+1)\, z^n \right) \phi(z)$$

We have

$$[Q_\epsilon, \phi(w)] = \frac{1}{2\pi i} \oint_{|z|>|w|} dz\, \epsilon(z)\, T(z)\, \phi(w) - \frac{1}{2\pi i} \oint_{|z|<|w|} dz\, \epsilon(z)\, \phi(w)\, T(z)$$

$$= \frac{1}{2\pi i} \left( \oint_{|z|>|w|} - \oint_{|z|<|w|} \right) dz \;\; \epsilon(z) \;\; R\left(T(z)\, \phi(w)\right)$$



$$= \frac{1}{2\pi i} \oint_{C_w} dz \;\; \epsilon(z) \;\; R\left( T(z)\, \phi(w) \right)$$

OPERATOR PRODUCT EXPANSION

$$R(T(z)\, \phi(w)) = \sum_n (z-w)^n \, O_n(w)$$

We get required result if

$$R(T(z)\, \phi(w)) = \frac{h\, \phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w} + \text{regular}$$

We have already seen that the quantum algebra of the $L_m$'s in general requires a central term

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{m+n,0}$$

this means that $T(z)$ doesn't transform as an $h=2$ conformal field (as one would naively expect), but according to

$$\delta_\epsilon T(z) = (\epsilon \partial + 2 \partial\epsilon) T(z) + \frac{c}{12} \partial^3 \epsilon \quad = [Q_\epsilon , T(z)]$$

or, the corresponding finite transformation is

$$T(z) \to T(w) \left(\frac{\partial w}{\partial z}\right)^2 + \frac{c}{12} \{w, z\}$$

where

$$\{w, z\} = \frac{\partial w \; \partial^3 w - \frac{3}{2}(\partial^2 w)^2}{(\partial w)^2}$$

is the Schwarzian derivative.

Note that $\partial^3 \epsilon(z) = 0$ for

$$\epsilon(z) = \underset{\substack{\uparrow \\ L_{-1}}}{\epsilon_{-1} z^2} + \underset{\substack{\uparrow \\ L_0}}{\epsilon_0 z} + \underset{\substack{\uparrow \\ L_1}}{\epsilon_1}$$

and that $\{w, z\} = 0$ iff $w = \dfrac{az+b}{cz+d}$, $ad - bc = 1$.

Note also that for $z = e^{iw}$ we have

$$\{z, w\} = \frac{(e^{iw})^4 - \frac{3}{2}(e^{iw})^4}{(ie^{iw})^2} = -\frac{1}{2} \left(\frac{\partial z}{\partial w}\right)^2$$

hence

$$T^{cyl} (dw)^2 = \left(T^{plane}(z) - \frac{c}{24}\right)(dz)^2$$

Claim $R\left(T(z)\,T(w)\right) = \dfrac{c/2}{(z-w)^4} + \dfrac{2T(w)}{(z-w)^2} + \dfrac{\partial T(w)}{(z-w)} + \text{reg.}$

$\uparrow$

only terms possible based on $L_0, L_\pm$ invariance !

proof:

$$[L_m, L_n] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} R\left(T(z)\,T(w)\right)$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left( \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right)$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \Big( \frac{c}{2} \frac{1}{3!} m(m^2-1) w^{m+n-1} + 2(m+1) w^{m+n} T(w)$$
$$+ w^{m+n+2} \partial T(w) \Big)$$

$$= \frac{c}{12} m(m^2-1)\,\delta_{m+n,0} + (m-n) \underbrace{\oint_{C_0} \frac{dw}{2\pi i} w^{m+n+1} T(w)}_{L_{m+n}}$$