We ogogog

Supersymmetric point particle

$$S' = -\int d\tau \left\{ -\frac{\dot{x}^2}{2e} + \frac{i}{e}\dot{x}^\mu \psi_\mu \chi - i\psi^\mu \dot{\psi}_\mu \right\}$$

$$\psi^\mu \psi^\nu = -\psi^\nu \psi^\mu, \qquad \psi^\mu \chi = -\chi \psi^\mu$$

Reparametrization invariance :
$$\begin{cases} \delta x^\mu = \xi \dot{x}^\mu & \delta \psi^\mu = \xi \dot{\psi}^\mu \\ \delta e = \frac{d}{d\tau}(\xi e) & \delta \chi = \frac{d}{d\tau}(\xi \chi) \end{cases}$$

Grassmann variable

Supersymmetry :
$$\begin{cases} \delta x^\mu = i\psi^\mu \epsilon \\ \delta e = i\chi \epsilon \\ \delta \chi = \dot{\epsilon} \\ \delta \psi^\mu = \frac{1}{2e}(\dot{x}^\mu + i\chi\psi^\mu)\epsilon \end{cases}$$

gauge $\quad e=1, \quad \chi = 0 \qquad$ (after that only global susy remains)

World sheet supergravity

Introduce zweibein e^a_α by $\quad h_{\alpha\beta} = \eta_{ab} e^a{}_\alpha e^b{}_\beta$

$$S' = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left\{ h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right.$$
$$\left. + 2\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \partial_\beta x_\mu + \frac{1}{2}\bar{\psi}_\mu \psi^\mu \bar{\chi}^\alpha \rho^\beta \rho^\alpha \chi_\beta \right\}$$

where $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ "2D gamma-matrices".

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\bar\psi = \psi^\dagger \rho^0 \qquad \bar\chi = \chi^\dagger \rho^0$$

$i\rho$'s are real.
take components of ψ, χ real (Majorana spinors)

<u>local SUSY</u>

$$\begin{cases} \delta X^\mu = \bar\epsilon \psi^\mu \\ \delta \psi^\mu = -i \rho^\alpha \epsilon \, (\partial_\alpha X^\mu - \bar\psi^\mu \chi_\alpha) \\ \delta e^a_\alpha = -2i \bar\epsilon \rho^a \chi_\alpha \\ \delta \chi_\alpha = \nabla_\alpha \epsilon \end{cases}$$

<u>Weyl invariance</u>

$$\delta X^\mu = 0 , \quad \delta \psi^\mu = -\tfrac{1}{2}\Lambda \psi$$

$$\delta e^a_\alpha = \Lambda e^a_\alpha \qquad \delta \chi_\alpha = \tfrac{1}{2}\Lambda \chi_\alpha$$

<u>Super Weyl invariance</u>

$$\delta \chi_\alpha = i\rho_\alpha \eta \qquad \delta e^a_\alpha = \delta\psi^\mu = \delta X^\mu = 0$$

(requires $\rho^\alpha \rho^\beta \rho_\alpha = 0$)

Together this gives a superconformal field theory.

Can use $\xi^\mu, \omega^{ab}, \Lambda$ to set $e^a_\alpha = \delta^a_\alpha$

ϵ, η " " $\chi_\alpha = 0$

What remains are eqn of motion of e and X or constraints

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2} \bar{\psi}^\mu e_{(\alpha} \partial_{\beta)} \psi_\mu - (\text{trace}) = 0$$

$$J_\alpha = \frac{1}{2} e^\beta e_\alpha \psi^\mu \partial_\beta X_\mu = 0$$

weighted symmetrization

$$S' = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left[\partial_\alpha X^\mu \partial^\alpha X_\mu - i \bar{\psi}^\mu e^\alpha \partial_\alpha \psi_\mu \right]$$

$$\psi = \psi_A = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$$

Dirac equation $\quad i e^\alpha \partial_\alpha \psi^\mu = 0$

or $\quad \begin{pmatrix} 0 & \partial_0 - \partial_1 \\ -\partial_0 \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} = 0$

recall, in terms of $\sigma^\pm = \tau \pm \sigma \qquad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

we have $\quad \partial_- \psi_+^\mu = 0, \quad \partial_+ \psi_-^\mu = 0$

supersymmetry : $\begin{cases} \delta X^\mu = \bar{\epsilon} \psi^\mu \\ \delta \psi^\mu = -i e^\alpha \epsilon \, \partial_\alpha X^\mu \end{cases}$

Action for fermion $\sim -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[-2 \left(\psi_-^\mu \; \psi_+^\mu \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_{-\mu} \\ \psi_{+\mu} \end{pmatrix} \right.$

$$\frac{i}{2\pi\alpha'} \int d^2\sigma \left[\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu} \right]$$

Boundary conditions: (open superstring)

$$\int d\tau \left(\psi^\mu_+ \, \delta\psi_{+\mu} - \psi^\mu_- \, \delta\psi_{-\mu} \right) \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

Hence $\psi^\mu_+ = \pm \psi^\mu_-$ at end-points

R: $\psi^\mu_+(\tau,0) = \psi^\mu_-(\tau,0)$, $\psi^\mu_+(\tau,\pi) = \psi^\mu_-(\tau,\pi)$

NS. $\psi^\mu_+(\tau,0) = \psi^\mu_-(\tau,0)$, $\psi^\mu_+(\tau,\pi) = -\psi^\mu_-(\tau,\pi)$

(same for all directions μ, to preserve D-dim'l rotational invariance)

R:

$$\psi^\mu_+(\tau,\sigma) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d^\mu_n \, e^{-in(\tau+\sigma)}$$

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\alpha'} \sum_{n\in\mathbb{Z}} d^\mu_n \, e^{-in(\tau-\sigma)}$$

NS

$$\psi^\mu_+(\tau,\sigma) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{-ir(\tau+\sigma)}$$

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{-ir(\tau-\sigma)}$$

For closed strings we can have periodic (R) or antiperiodic (NS)

boundary conditions for left/right movers independently, i.e.

RR	d^μ_m , $\tilde{d}^\nu_n$	$m, n \in \mathbb{Z}$
R.NS	d^μ_m , $\tilde{b}^\nu_s$	$m, s+\frac{1}{2} \in \mathbb{Z}$
NS.R	b^μ_r , $\tilde{d}^\nu_n$	$r+\frac{1}{2}, n \in \mathbb{Z}$
NS.NS	b^μ_r , $\tilde{b}^\nu_s$	$r+\frac{1}{2}, s+\frac{1}{2} \in \mathbb{Z}$

Note also
$$\begin{cases} T_{\pm\pm} = \partial_\pm X^\mu \partial_\pm X_\mu + \frac{i}{2} \psi^\mu_\pm \partial_\pm \psi_{\pm\mu} \\ J_\pm = \psi^\mu_\pm \partial_\pm X_\mu \end{cases}$$

QUANTIZATION

Note canonical momentum $\pi^\mu_A = \frac{\delta S}{\delta(\partial_0 \psi^A_\mu)} = \frac{i}{4\pi\alpha'} \psi^\mu_A$

$$\Rightarrow \{ \psi^\mu_A , \psi^\nu_B \} = (2\pi\alpha') \, \delta_{AB} \, \eta^{\mu\nu} \, \delta(\sigma - \sigma')$$

$$\Rightarrow \{ b^\mu_r , b^\nu_s \} = \eta^{\mu\nu} \delta_{r+s,0}$$

$$\{ d^\mu_m , d^\nu_n \} = \eta^{\mu\nu} \delta_{m+n,0}$$

(or, in terms of OPE's $\quad R(\psi(z)\psi(w)) = \frac{1}{z-w}$)

(R) $L_m = \frac{1}{2} \sum :\alpha_{-n} \alpha_{m+n}: + \frac{1}{2} \sum_n (n + \frac{1}{2} m) : d_{-n} d_{m+n} :$

(NS) $L_m = \quad '' \quad + \frac{1}{2} \sum_r (r + \frac{1}{2} m) : b_{-r} b_{m+r} :$

(R) $F_m = \sum_n \alpha_{-n} \cdot d_{m+n}$

(NS) $G_r = \sum_n \alpha_{-n} \cdot b_{n+r}$

Super Virasoro algebra (NS) $c = D + \frac{D}{2} = \frac{3D}{2}$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{8} m(m^2-1) \delta_{m+n,0}$$

$$[L_m, G_r] = (\tfrac{1}{2}m - r) G_{m+r}$$

$$\{G_r, G_s\} = 2 L_{r+s} + \frac{D}{2}(r^2 - \tfrac{1}{4}) \delta_{r+s,0}$$

Physical states :

$$\begin{cases} G_r |\phi\rangle = 0 & r > 0 \\ L_n |\phi\rangle = 0 & n > 0 \end{cases}$$

$$(L_0 - a)|\phi\rangle = 0$$

+ Similar in R-sector (find $D=10$, $a = \frac{1}{2}$ for NS-sector in no-ghost theorem and $a=0$ in R-sector)

<u>Spectrum of open superstring</u>

$$(NS) \qquad M^2 = \frac{1}{\alpha'}\left(\sum_{m \geq 1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r} \cdot b_r - \frac{1}{2}\right)$$

$$(R) \qquad M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{n=0}^{\infty} n\, d_{-n} \cdot d_n\right)$$

(NS) $\quad |k\rangle \qquad M^2 = -\frac{1}{2\alpha'}$

$\quad b^{\mu}_{-\frac{1}{2}} |k\rangle \qquad M^2 = 0$

$\vdots$

In (R) sector, ground state is massless. It is however degenerate as we can act on it by d_0^μ. Note $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$

i.e. $d_0^\mu \sim \Gamma$ - matrix in D-dimensions

In general (for even D) Γ-matrices $2^{(D/2)}$-dimensional.

Hence in (R) sector, ground state is a 32-dim'l spinor.

$$d_0^\mu |\alpha\rangle_R = \frac{1}{\sqrt{2}} \Gamma^\mu_{\alpha\beta} |\beta\rangle_R$$

Problems

(i) still a tachyon in NS sector

(ii) no space-time supersymmetry

e.g. at massless level $b^\mu_{-\frac{1}{2}} |k\rangle_{NS}$ 8 d.o.f

$|\alpha\rangle_R$ 16 d.o.f

(and $\alpha^\mu_{-1} |\ \rangle_{NS}$ does not have partner)

Do a projection. (GSO-projection)

$\Gamma_{11} = \Gamma^0 \ldots \Gamma^9$ $\quad \{\Gamma_{11}, \Gamma^\mu\} = 0$, $(\Gamma_{11})^2 = 1$

we have projection operators $\Pi_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$

Choose Majorana-Weyl spinors (possible in D=10) by requiring $\Gamma_{11} = 1$.

Also have "fermion counting operator"

$$(-1)^F X^\mu = X^\mu, \qquad (-1)^F \psi^\mu = -\psi^\mu$$

Specifically,

$$(-1)^F_{NS} = -(-1)^{\sum_{r=\frac{1}{2}}^{\infty} b_{-r}\cdot b_r}$$

$$(-1)^F_R = \Gamma_{11}\,(-1)^{\sum_{n=1}^{\infty} d_{-n}\cdot d_n}$$

and require $(-1)^F |\phi\rangle = |\phi\rangle$

FREE FERMION PARTITION FUNCTION

NS $\{b_r, b_s\} = \delta_{r+s,0}$

$$\mathrm{Tr}\, q^{L_0} = \prod_{r=\frac{1}{2}}^{\infty} (1+q^r) \qquad \left(= \prod_r \sum_{N_r=0}^{1} (q^{rN_r}) \right)$$

R $\{d_m, d_n\} = \delta_{m+n,0}$

$$\mathrm{Tr}\, q^{L_0} = \# \prod_{n=1}^{\infty} (1+q^n)$$

degeneracy ground state.

STRING PARTITION FUNCTION

$$Z_{NS}(\tau) = \frac{1}{2}\,\mathrm{Tr}\left(q^{L_0-\frac{1}{2}} \left(1 + (-1)^F\right)\right)$$

$$= \frac{1}{2\sqrt{q}}\left[\prod_{m=1}^{\infty} \left(\frac{1+q^{m-\frac{1}{2}}}{1-q^m}\right)^8 - \prod_{m=1}^{\infty} \left(\frac{1-q^{m-\frac{1}{2}}}{1-q^m}\right)^8 \right]$$

$$Z_R(\tau) = \frac{1}{2} \mathrm{Tr}\, q^{L_0} \Gamma_{11} (1 + (-1)^F) = 8 \prod_{m \geq 1} \left(\frac{1+q^m}{1-q^m} \right)^8$$

$$Z_{NS} = Z_R \quad (\text{Jacobi } '1829)$$

In closed superstring we also need GSO projection Choice of

$\Gamma_{11} = \pm 1$ in left/right

Opposite chirality IIA
Same chirality IIB

Massless states

	ϕ		$B_{\mu\nu}$		$g_{\mu\nu}$	
NS - NS	dilaton	,	anti-symmetric rank 2	,	symmetric traceless	
	1	+	28	+	35	= 64

NS - R spin-$\frac{1}{2}$ fermion , spin $\frac{3}{2}$ fermion (gravitino)
8 + 56 = 64

R - NS "

R R IIA vector + anti-symm. rank 3 tensor
8 56 = 64

IIB scalar + rank 2 antisymm + rank 4 antiselfd. antisymm
1 + 28 + 35
= 64