we aqujor

Path integral quantization

$$Z = \int \mathcal{D}h \, \mathcal{D}X \; e^{-S[h,X]}$$

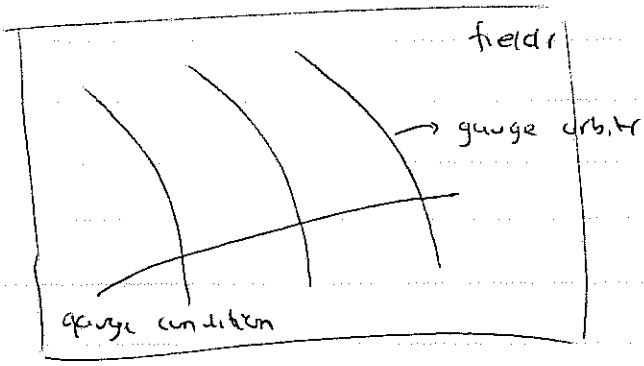
$$S[h,X] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{h} \, h^{\alpha\beta} \, \partial_\alpha X^\mu \, \partial_\beta X_\mu$$

(Euclidean formulation)

Path integral is invariant under diffeomorphisms & Weyl transformations, so we want to divide out by these, ie we want to divide by

$$\mathrm{Vol}(\mathrm{Diff}) \times \mathrm{Vol}(\mathrm{Weyl})$$

or, alternative choose a gauge, and only integrate each over slice.



Using, using diffeomorphisms, we can choose $h^{\alpha\beta} = e^{\phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

or in complex (light-cone) coordinates $h_{zz} = 0$, $h_{\bar{z}\bar{z}} = 0$ $h_{z\bar{z}} = \frac{1}{2}e^{\phi}$

So, to get rid of diffeomorphism invariance we are tempted to insert

$\delta(h_{zz})\,\delta(h_{\bar{z}\bar{z}})$ in path integral. This is incorrect.

FINITE DIMENSIONAL EXAMPLE

Consider $I = \int d^2\vec{x} \quad F(|\vec{x}|)$

has rotation invariance. By going to polar coordinates (θ is coordinate along the 'gauge orbit' we have

$$I = \int_0^{2\pi} d\theta \int_0^{\infty} dr \quad r \, F(r)$$

Alternative, we may impose a gauge, say $y = 0$. Clearly, inserting $\delta(y)$ gives wrong answer

$$I \neq \int d^2x \quad F(|\vec{x}|) \; \delta(y) = \int dx \quad F(|x|)$$

we're missing the Jacobian!

Note that $\int d\theta \quad \delta(f(\theta)) \; |f'(\theta)| = 1$

Denote by ${}^{\theta}\vec{x}$ the rotation of $\vec{x}$ under θ ie $\begin{pmatrix} {}^{\theta}x \\ {}^{\theta}y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

then $\int d\theta \; \delta({}^{\theta}y) \left| \frac{d\,{}^{\theta}y}{d\theta} \right| = 1$

Insert this in integral

$$I = \int d\theta \int d^2\vec{x} \quad \delta({}^{\theta}y) \left| \frac{d\,{}^{\theta}y}{d\theta} \right| \; F(|\vec{x}|)$$

$$= \int d\theta \int d^2\vec{x} \quad \delta(y) \; |x| \; F(|\vec{x}|)$$

New variables $\vec{x}' = {}^{\theta}\vec{x}$

$$= \int d\theta \int dx \quad |x| \; F(|x|)$$

where we have used

$$\frac{d\,{}^{\theta}y}{d\theta} = \frac{d}{d\theta}\left(-x\sin\theta + y\cos\theta \right) = -\left(x\cos\theta + y\sin\theta \right) = -x'$$

(Note, since we will put $y=0$, it is sufficient to look at infinitesimal transformation of y at $y=0$; ie $y' = y - x\theta$ $\quad \frac{dy'}{d\theta} = -x$)

Back to our example. We insert

$$1 = \int \mathcal{D}g \; \delta({}^{g}h_{zz})\, \delta({}^{g}h_{\bar{z}\bar{z}}) \, \det\left(\frac{\delta\, {}^{g}h_{zz}}{\delta g}\right) \det\left(\frac{\delta\, {}^{g}h_{\bar{z}\bar{z}}}{\delta g}\right)$$

go to new variables ${}^{g}h_{\alpha\beta}$ and find

$$Z = \int \mathcal{D}g \int \mathcal{D}\phi \int \mathcal{D}h_{zz}\, \mathcal{D}h_{\bar{z}\bar{z}}\, \mathcal{D}X \; \delta(h_{zz})\, \delta(h_{\bar{z}\bar{z}})$$
$$\det\left(\frac{\delta h_{zz}}{\delta g}\right) \det\left(\frac{\delta h_{\bar{z}\bar{z}}}{\delta g}\right) \; e^{-S[h,X]}$$

Now $\delta h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$ under diffeomorphism

hence $\delta h_{zz} = 2\nabla_z \xi_z$

$$\delta h_{\bar{z}\bar{z}} = 2\nabla_{\bar{z}} \xi_{\bar{z}}$$

where $\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^{\gamma}_{\alpha\beta}\, \xi_\gamma$

$$\Gamma^{\gamma}_{\alpha\beta} = \tfrac{1}{2} h^{\gamma\delta}\left(\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta} \right)$$

In the gauge $h_{zz} = h_{\bar{z}\bar{z}} = 0 \quad h_{z\bar{z}} = \frac{1}{2} e^{\phi}$

we have $\Gamma^{z}_{zz} = \partial_z \phi \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}} \phi$

eg. $\Gamma^{z}_{zz} = \frac{1}{2} h^{z\bar{z}} \left(2\partial_z h_{z\bar{z}} - \partial_{\bar{z}} h_{zz} \right)$ (using $h^{z\bar{z}} = 2e^{-\phi}$

$$= h^{z\bar{z}} \partial_z h_{z\bar{z}} = \partial_z \phi$$

So $\det \left(\frac{\delta h_{zz}}{\delta \xi_z} \right) \sim \det (\nabla_z)$

$$\det \left(\frac{\delta h_{\bar{z}\bar{z}}}{\delta \xi_{\bar{z}}} \right) \sim \det (\nabla_{\bar{z}})$$

How do we calculate these determinants ?

Note, for Gaussian integrals

$$\int dx_1 .. dx_n \; e^{-\vec{x} A \vec{x}} = \pi^{n/2} (\det A)^{-\frac{1}{2}}$$

eg. diagonalize and use $\int dx \; e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$.

or, in the case of complex variables

$$\int dz_1 .. d\bar{z}_1 \dots \; e^{-\bar{z} A z} \sim (\det A)^{-1}$$

To get $(\det A)^{+1}$ we need Grassmann variables (anticommuting variables)

eg $\bar{\theta}, \theta$ st $\bar{\theta}\theta + \theta\bar{\theta} = 0$ and define Berezin integral
$\theta^2 = 0, \; \bar{\theta}^2 = 0$

$$\int d\theta \, . \, 1 = 0 \qquad \int d\theta \; \theta = 1$$

then eg

$$\int d\theta\, d\bar\theta \; e^{\bar\theta M \theta} = \int d\theta\, d\bar\theta \; (1 + \bar\theta M \theta) = M$$

or more generally

$$\int d\theta_1 \dots d\theta_n \, d\bar\theta_1 \dots d\bar\theta_n \; e^{\bar\theta \cdot A \cdot \theta} = \det A.$$

Fadeev-Popov ghosts

I.e. $$\det(\nabla_z) = \int \mathcal{D}c^{\bar z}\, \mathcal{D}b_{\bar z \bar z} \; e^{-\frac{1}{\pi}\int d^2 z \; c^{\bar z}\, \nabla_z\, b_{\bar z \bar z}}$$

$$\det(\nabla_{\bar z}) = \int \mathcal{D}c^{z}\, \mathcal{D}b_{zz} \; e^{-\frac{1}{\pi}\int d^2 z \; c^{z}\, \nabla_{\bar z}\, b_{zz}}$$

and $$Z = \int \mathcal{D}\phi \int \mathcal{D}X\, \mathcal{D}c\, \mathcal{D}b \; e^{-S'[X,b,c]}$$

with

$$S'[X,b,c] = \frac{1}{2\pi}\int d^2 z \left(\partial_z X \cdot \partial_{\bar z} X + \overbrace{2 c^{\bar z} \nabla_z b_{\bar z \bar z} + 2 c^{z} \nabla_{\bar z} b_{zz}}^{S_{FP}} \right)$$

or, written for an arbitrary worldsheet metric

$$S_{FP} = \frac{1}{\pi} \int d^2\sigma \; \sqrt{h}\; h^{\alpha\beta}\, c^{\gamma}\, \nabla_\alpha\, b_{\beta\gamma}$$

where c^α contravariant, $b_{\alpha\beta}$ covariant, symmetric traceless

In the usual manner

$$T^{(bc)}_{\alpha\beta} = \frac{1}{2} c^{\gamma}\, \nabla_{(\alpha}\, b_{\beta)\gamma} + 2\,(\nabla_{(\alpha} c^{\gamma})\, b_{\beta)\gamma} - \text{trace}$$

and, in complex coordinates

$$T_{zz} = c^z \partial_z b_{zz} + 2\, \partial_z c^z\, b_{zz}$$

$$= c\,\partial b + 2\,\partial c\, b \qquad \text{for short} \qquad \begin{pmatrix} c^z = c \\ b_{zz} = b \\ c^{\bar z} = \bar c \\ b_{\bar z \bar z} = \bar b \end{pmatrix}$$

and similarly for $T_{\bar z \bar z}$.

<u>Quantization</u>

In conformal gauge $\nabla_z b_{\bar z \bar z} = \partial_z b_{\bar z \bar z}$ etc so equations of motion give $\partial \bar b = \partial \bar c = 0$ $\quad \bar\partial b = \bar\partial c = 0$

Hence $c(z) = \sum c_n z^{-n+1}$ $\quad (h = -1)$

$b(z) = \sum b_n z^{-n-2}$ $\quad (h = 2)$

and canonical quantization gives

$\{c_m, b_n\} = \delta_{m+n,0}$

anticommutators

or, equivalently $R(c(z)\, b(w)) = \frac{1}{z-w} + \ldots$

$R(b(z), c(w)) = \frac{1}{z-w} + \ldots$

A little calculation gives, for arbitrary λ, that

$$T(z) = \lambda\, \partial c\, b - (1-\lambda)\, c\, \partial b = -c\partial b + \lambda\, \partial(cb)$$

satisfies Virasoro algebra with $c = 1 - 3(1-2\lambda)^2$

and (b, c) are primary fields with dimensions $(\lambda, 1-\lambda)$

Taking $\lambda = 2$ gives $c = -26$

$\Rightarrow$ for $D = 26$ no conformal anomaly and we can forget about

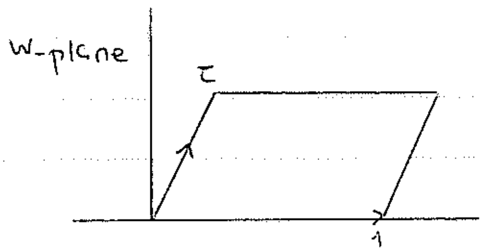
the $\int \mathcal{D}\phi$ integral (giving Vol (Weyl))

LOOP AMPLITUDES

In the case of a 1-loop diagram (torus) we have remaining

"moduli" or not all tori are conformally equivalent.

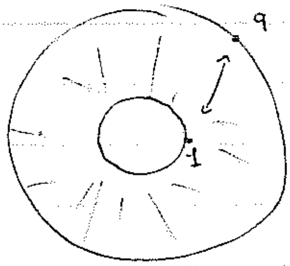
Using translation, rotation and scaling, we can put torus in standard form



$w \sim w+1$

$w \sim w+\tau$

or in terms of $z = e^{2\pi i w}$ $\qquad z \sim qz \qquad q = e^{2\pi i \tau}$



looking at evolution operator we need to insert twist

$$2\pi i \tau_1 P - 2\pi \underbrace{\tau_2}_{T} H = 2\pi i L_0 \tau - 2\pi i \bar{L}_0 \bar{\tau} + \frac{\pi \tau_2}{24}$$

T duration of propagation

using $P = L_0 - \bar{L}_0$

$H = L_0 + \bar{L}_0 - \frac{1}{12}$ $\quad$ ✓ conformal anomaly

The integrand of the path integral gives (for a single boson)

$$Z(\tau,\bar{\tau}) = (q\bar{q})^{-1/24} \, \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0}\, \bar{q}^{\bar{L}_0}\right)$$

↑ character

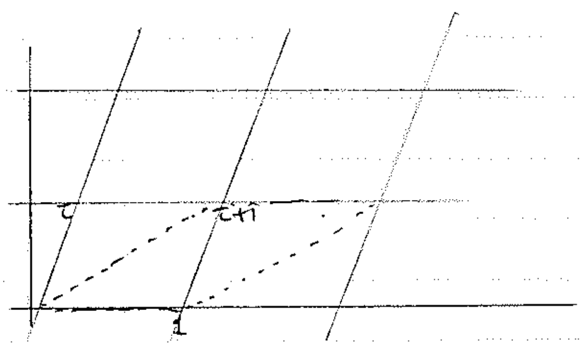
$$Z(\tau,\bar{\tau}) = (q\bar{q})^{-1/24} \int dk \; e^{-\pi\tau_2\alpha' k^2} \prod_n \sum_{\substack{N_n=0 \\ \bar{N}_n=0}}^{\infty} q^{nN_n}\, \bar{q}^{n\bar{N}_n}$$

$$\sim (4\pi^2\alpha'\tau_2)^{-\frac{1}{2}} \left| \prod_{n\geq 1} \frac{q^{-1/24}}{(1-q^n)} \right|^2$$

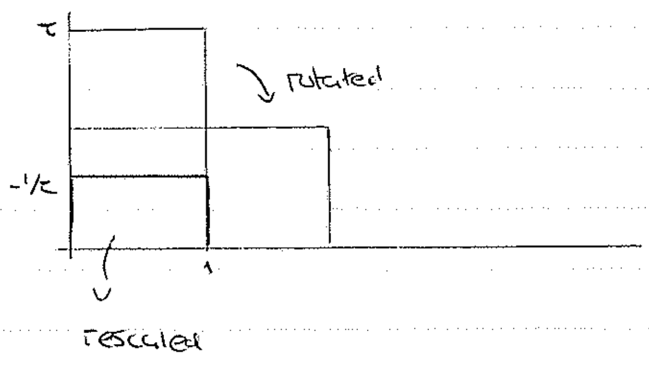
Remaining integral over moduli is $\displaystyle\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}$

↑ careful analysis of path integral

MODULAR INVARIANCE



$\tau \to \tau+1$



$\tau \to -\frac{1}{\tau}$.

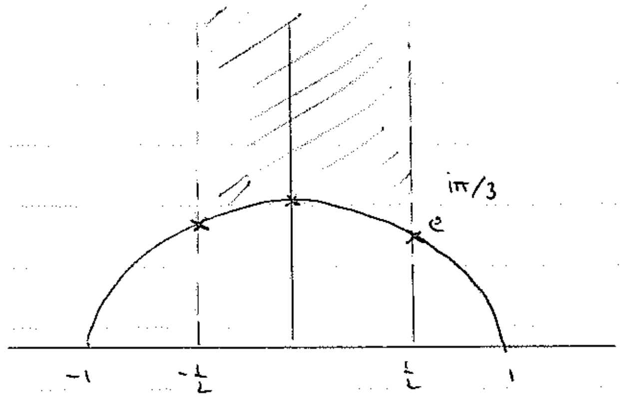
$T: \tau \to \tau+1$ generate modular group of torus
$S: \tau \to -\frac{1}{\tau}$

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

i.e. $ad - bc = 1$, $a,\ldots,d \in \mathbb{Z}$

($(ST)^3 = S^2 = 1$ are relations)

So, instead of integrating over $\mathcal{F}$ (Teichmüller space), we should integrate over $\mathcal{F}/SL(2,\mathbb{Z})$ (Moduli space)



x = orbifold points

Note that $d\tau' = \dfrac{d\tau}{(c\tau+d)^2}$ and $\operatorname{Im}\tau' = \dfrac{\operatorname{Im}\tau}{|c\tau+d|^2}$

hence $\dfrac{d^2\tau}{(\operatorname{Im}\tau)^2}$ is modular invariant, and hence the integrand should be as well. Indeed

$$Z(\tau,\bar\tau) \sim \frac{1}{(\operatorname{Im}\tau)^{12}} \left(\eta(\tau)\,\bar\eta(\bar\tau)\right)^{-24}$$

is modular invariant.