Dissipative Quantum Systems: Exact Results

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RSPhysSE, September 06, 2007

[based on works with S. Lukyanov and A. Zamolodchikov]

Brownian motion in a tilted washboard potential

$$M\ddot{q} + \eta\dot{q} + \frac{\partial U(q)}{\partial q} = \xi(t)$$

$\xi(t)$ is random force (thermal noise),

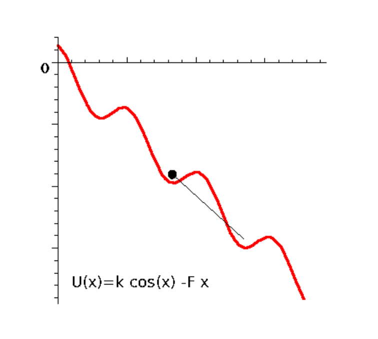
$< \xi(t)\xi(t') >= 2\eta T\delta(t-t')$, $\quad T = \text{temperature}$

η is friction coefficient (cf. Einstein relation).

$$U(q) = -\kappa\cos q - Vq$$

Parameters in the problem (quantum case):

$$\kappa, V, T, \hbar$$



Applications in condensed matter physics

- Diffusion of a charged particle in metals
- Resistively shunted (ultra-small) Josephson junctions
- Point contact current in the fractional quantum Hall effect.

THE QUANTUM MODEL INVENTED BY SOLID STATE PHYSICISTS IN 1980's

Brownian motion: classical theory

Langevin equation:

$$M\ddot{q} + \eta\dot{q} + \frac{\partial U(q)}{\partial q} = \xi(t), \qquad U(q) = -\kappa\cos q - Vq$$

$$\dot{q} = \kappa\sin q + V + 2\pi\xi(t), \quad M \to 0, \quad \eta = 1/(2\pi), \quad \text{(over-dumped regime)}$$

$$J(V) = <\dot{q}>_{t\to\infty} \quad \text{(mobility)}$$

For $T = 0$ the answer is $J = \sqrt{V^2 - \kappa^2}$

Fokker-Planck equation [Stratonovich (1967)]

$$\partial_t P = 2\pi T\ \partial_q \{ (-\nu + \sqrt{2}X\ \sin q)\ P + \partial_q P \}\ , \quad \nu = \frac{V}{2\pi T}, \qquad \sqrt{2}X = \frac{\kappa}{T}$$

$$P(z) = \mathcal{N}^{-1}\ P_0(z)\ , \quad P_0(z) = e^{\sqrt{2}X\cos(z)+\nu z}\int_z^{z+2\pi}\frac{dy}{2\pi}\ e^{-\sqrt{2}X\cos(y)-\nu y}, \quad \mathcal{N} = \int_0^{2\pi} dz\ P_0(z)$$

$$J = \frac{2T\sinh(\pi\nu)}{I_{i\nu}(\sqrt{2}X)\ I_{-i\nu}(\sqrt{2}X)} = i\pi\ T\,\kappa\,\partial_\kappa\Big(\log I_{i\nu}(\frac{\kappa}{T}) - \log I_{-i\nu}(\frac{\kappa}{T})\Big),$$

$I_\rho(x)$ is the modified Bessel function.

Classical field theory: Boundary sine-Gordon model

$$\mathcal{A}_{BSG} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left(\Phi_t^2 - \Phi_x^2 \right) + \kappa \int_{-\infty}^{\infty} dt \, \cos\left(\Phi_B + Vt\right),$$

$\Phi_B(t) = \Phi(x,t)|_{x=0}$ — boundary field,

$$(\partial_t^2 - \partial_x^2)\Phi = 0, \quad \Phi = f(t-x) + g(t+x), \quad x < 0$$

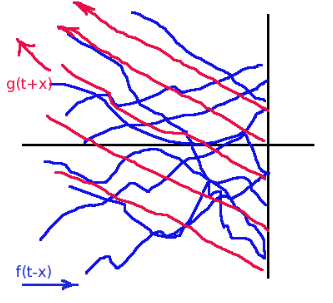
$$\partial_x \Phi|_{x=0} = -\kappa \sin(\Phi_B + Vt)$$

$$\Phi_B(t) = f(t) + g(t), \quad \partial_x\Phi|_{x=0} = g'(t) - f'(t)$$

$$\dot{\Phi}_B = -\kappa \sin\Big(\Phi_B(t) + Vt\Big) + 2f'(t)$$

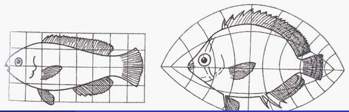
$$\dot{q} = \kappa \sin q + V + 2\pi\xi(t), \qquad z(t) = \Phi_B + Vt$$

$2f'(t)$ — random force.



$$z \to z' = w(z), \quad \bar{z} \to \bar{z}' = \overline{w}(\bar{z}),$$

$w, \overline{w}$ — arbitrary analytic functions



Dissipative Quantum Mechanics: Caldeira-Leggett model

$$\mathcal{H} = \frac{P^2}{2M} + U(q) + \sum_a \frac{p_a^2}{2m_a} + \frac{1}{2}\sum m_a\omega_a^2(x_a + \frac{\lambda_a q}{m_a\omega_a^2})^2$$

$$S_{eff} = \int_{-\infty}^{\infty}\Big(\frac{\dot{q}^2}{2M} - U(q)\Big)dt - \frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\alpha(t-t')[q(t)-q(t')]^2$$

$$\alpha(t-t') = \frac{1}{2\pi}\int_0^{\infty} J(\omega)e^{i\omega|t-t'|}d\omega, \qquad J(\omega) = \frac{\pi}{2}\sum_a \frac{\lambda_a^2}{m_a\omega_a}\delta(\omega-\omega_a)$$

Ohmic dissipation : $\quad J(\omega) = \eta\omega, \quad \alpha(t-t') = \dfrac{\eta}{2\pi(t-t')^2}$

$$S_0 = \int \frac{d\omega}{4\pi} D_\omega^{-1}|q_\omega|^2, \quad D(t) = \int_0^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega\, t}}{M\omega^2 + J(\omega)} = -\frac{1}{\pi\eta}\log(\eta t/M),\, t \gg 1$$

Equilibrium systems: Matsubara imaginary time approach

$$Z = \int e^{-S_E/\hbar}\mathcal{D}q(\tau),\; S_E = \int_0^{\beta}\Big(\frac{\dot{q}^2}{2M} + U(q)\Big)d\tau + \frac{\pi\eta}{4\beta^2}\int_0^{\beta} d\tau \int_0^{\beta} d\tau' \frac{(q(\tau)-q(\tau'))^2}{\sinh^2(\pi T(\tau-\tau'))},$$

QFT approach for Caldeira-Leggett model. Callan-Thorlacius (1990)

$$\mathcal{A}_{BSG} = \frac{1}{4\pi g}\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\,\left(\Phi_t^2 - \Phi_x^2\right) + \frac{\kappa}{g}\int_{-\infty}^{\infty} dt\, \cos\left(\Phi_B + Vt\right), \quad g = 1/\eta$$

Missing inertia term $\dot{q}^2/2M$ replaced by UV cutoff $\Lambda \sim M$. $U(q) = -\kappa\cos q - Vq$.

RG analysis: Fisher & Zwerger (1985)

- Cutoff dependence $\kappa = [\mathrm{mass}]^{1-\mathrm{g}}, \quad \kappa(\mu) = \kappa(\Lambda)\,(\mu/\Lambda)^{\mathrm{g}-1}$
- for $\eta > 1$ or $g < 1$ (which is strong dissipation) the particle is localized at $T = 0$. Dimensionless parameter κV^{g-1} — strong coupling (strong barrier) at small V.
- For weak dissipation $g > 1$ the particle is delocalized: band structure, etc. There is no universal low-temperature behavior. Results depend on the cutoff details.

Schwinger-Keldysh real-time perturbation theory

$$\mathbf{H}_0 = \frac{1}{4\pi g}\int_{-\infty}^{0} dx\Big(\Pi(x)^2 + \Phi_x^2\Big), \ \mathbf{H}_1 = -\frac{\kappa}{g}\cos(\Phi_B + Vt), \quad [\Pi(x), \Phi(x)] = -2i\pi g\delta(x - x')$$

$$\mathbf{P}(t) = e^{-i\mathbf{H}_0 t}\, \mathbf{S}(t,-\infty)\, \mathbf{P}_0\, \mathbf{S}(-\infty,t)\, e^{i\mathbf{H}_0 t}, \quad \mathbf{S}(t,t_0) = \mathcal{T}\exp\Big\{ -i\int_{t_0}^{t} d\tau\, \mathbf{H}_1^{(int)}(\tau)\Big\}$$

$$\mathbf{P}_0 = Z_0^{-1} e^{-g\mathbf{H}_0/T}, \quad < \mathbf{A}(t) >_{BSG} = \mathrm{Tr}_{\mathcal{F}}\big[\,\mathbf{P}(\mathrm{t})\,\mathbf{A}\,\big] =< \mathbf{S}(-\infty,\mathrm{t})\,\mathbf{A}^{(\mathrm{int})}(\mathrm{t})\,\mathbf{S}(\mathrm{t},-\infty) >_0$$

$$J =< \Phi_x(0,t) >_{BSG} = i\pi\kappa \ < \big(\mathbf{V}_+(t) - \mathbf{V}_-(t)\big) >_0, \quad < \ldots >_0 = \mathrm{Tr}[\ldots \mathbf{P}_0]$$

$$\mathbf{V}_\pm(t) = \mathbf{S}(-\infty,t)\, \exp\big\{ \pm i\Phi_B^{(int)}(t) \pm iVt\big\}\, \mathbf{S}(t,-\infty)\ ,$$

Perturbation series, thermalisation. Weiss et al (1995)

$$\langle\, \mathbf{V}_\sigma \,\rangle_{BSG} = -\frac{2\pi\sigma\, T}{\kappa\, \sin(\pi g)} \sum_{n=1}^{\infty} \lambda^{2n} \sum_{\sigma_1,\ldots,\sigma_{2n-1}=\pm 1} \Big(\prod_{j=1}^{2n-1} \frac{\sin(\pi g \eta_j)}{\sin(\pi g)}\Big)\, J(\sigma,\sigma_1,\ldots,\sigma_{2n-1}|\, p)\ ,$$

with $\lambda \sim \kappa T^{g-1}$, $p \sim iV/T$ and $\eta_j = \sum_{k=0}^{j-1} \sigma_k$

$$J_n(\{\sigma\}|\, p) = \int_{t_0}^{0} \Big(\prod_{k=1}^{2n-1} d\tau_k\Big) e^{-2p\sum_{j=1}^{2n-1} \sigma_j \tau_j} \prod_{0\le j<l\le 2n-1} \Big(2\sinh\Big(\frac{\tau_j - \tau_l}{2}\Big)\Big)^{2g\sigma_j\sigma_l}$$

where $\tau_0 = 0$, and $0 > \tau_1 > \tau_2 \ldots$,

Exact solution strategy. BLZ (1996-2001)

- Consider three equilibrium QFT with boundary interaction

$$\mathbf{H}_+ = \frac{1}{4\pi g}\int_{-\infty}^{0} dx\,\big(\Pi^2+\Phi_x^2\big) - V\,\mathbf{h} - \frac{\kappa}{2g}\,\big(\mathbf{a}_-\,e^{i\Phi_B}+\mathbf{a}_+\,e^{-i\Phi_B}\big)\,,$$

$$\big[\mathbf{h},\mathbf{a}_\pm\big]=\pm\mathbf{a}_\pm\,;\quad q\,\mathbf{a}_+\mathbf{a}_- - q^{-1}\,\mathbf{a}_-\mathbf{a}_+ = q-q^{-1},\quad q=e^{i\pi g}$$

$$\mathbf{H}_- = \frac{1}{4\pi g}\int_{-\infty}^{0} dx\,\big(\Pi^2+\Phi_x^2\big) + V\,\mathbf{h} - \frac{\kappa}{2g}\,\big(\mathbf{a}_+\,e^{i\Phi_B}+\mathbf{a}_-\,e^{-i\Phi_B}\big)$$

$$\mathbf{H}_{Kondo} = \frac{1}{4\pi g}\int_{-\infty}^{0} dx\,\big(\Pi^2+\Phi_x^2\big) + V\,\sigma_z - \frac{\kappa}{2g}\,\big(\sigma_+\,e^{i\Phi_B}+\sigma_-\,e^{-i\Phi_B}\big)$$

  where $\sigma_z,\sigma_+,\sigma_-$ are Pauli matrices.
- Using full power of Yang-Baxter integrability, extended to continuous QFT, prove the functional relation:

$$\mathbf{T}(\kappa)\,\mathbf{Q}_\pm(\kappa)=\mathbf{Q}_\pm(\kappa q)+\mathbf{Q}_\pm(\kappa q^{-1}),\quad \mathbf{T}=\mathrm{Tr}\,\mathrm{e}^{-\beta\mathrm{H}_{\mathrm{Kondo}}},\quad \mathbf{Q}_\pm=\mathrm{Tr}\,\mathrm{e}^{-\beta\mathrm{H}_\pm},$$

- Using real-time perturbation theory prove that (and similarly for $\langle\,\mathbf{V}_-\,\rangle_{BSG}$)

$$\langle\,\mathbf{V}_+\,\rangle_{BSG}=\langle\,\mathbf{W}_+\,\rangle_+=\langle\,\mathbf{W}_-\,\rangle_+,\quad \text{where}\quad \mathbf{W}_+=\mathbf{a}_-\,e^{i\Phi_B}\,;\quad \mathbf{W}_-=\mathbf{a}_+\,e^{-i\Phi_B}\,.$$

- We obtain

$$J = i\pi T\kappa\partial_\kappa \log \frac{\mathbf{Q}_+(\kappa)}{\mathbf{Q}_-(\kappa)}$$

in the classical limit $g \to 0$, $Q_\pm$ reduce to Bessel functions leading to Stratonovich's result

$$J = i\pi\, T\, \kappa\, \partial_\kappa \Big(\log I_{i\nu}(\frac{\kappa}{T}) - \log I_{-i\nu}(\frac{\kappa}{T}) \Big),$$

- Connect $\mathbf{T}$, $\mathbf{Q}_\pm$ with spectral theory of one-dimensional Schrödinger equation. Dorey-Tateo, BLZ (1998)

$$-\partial_y^2\, \tilde{\Psi} + \Big\{ e^{\frac{2y}{g}} - E\, e^{2y} \Big\}\, \tilde{\Psi} = \nu^2\, \tilde{\Psi}, \quad \nu = V/2\pi T, \quad E = -\frac{\kappa^2 \pi}{\Gamma(1+g)^2} (\pi T)^{2g-2}$$

$$\tilde{\Psi}(y) \to e^{i\nu y} + S(\nu, E)\, e^{-i\nu y} \quad y \to -\infty\ ,$$

- The answer for J is

$$J = V - 2\pi i T E \partial_E \log S(\nu, E)$$

- Strong-weak barrier duality (Schmit, Fisher-Zwerger, Saleur-Fendley, BLZ)

$$J(V, E, g) = -V - g^{-1} J(gV, E^{-1/g}, g^{-1})$$

Ultra-small shunted Josephson junctions

Basic theory. Josephson (1962)

$$I_s = I_c \sin\varphi, \quad \partial_t \varphi = 2eV/\hbar, \quad \varphi = \text{phase difference}$$

$$F = const - E_J \cos\varphi, \quad E_J = (\hbar I_c/2e), \quad E_J > kT$$

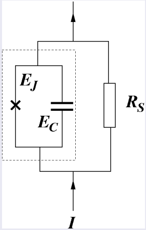
$$2ek/\hbar \sim 0.04\mu A K^{-1}, \quad \text{typical current} \quad 1\mu A - 10mA$$

Shunted junction

$$I = I_c \sin\varphi + V/R_S + CdV/dt$$

$$\frac{\hbar^2}{8E_C}\,\ddot{\phi} + \frac{\partial U_I(\phi)}{\partial\phi} = -\frac{\hbar\alpha}{2\pi}\,\dot{\phi}, \quad U_I(\phi) = -E_J\cos(\phi) - \frac{\hbar I}{2e}\,\phi\,,$$

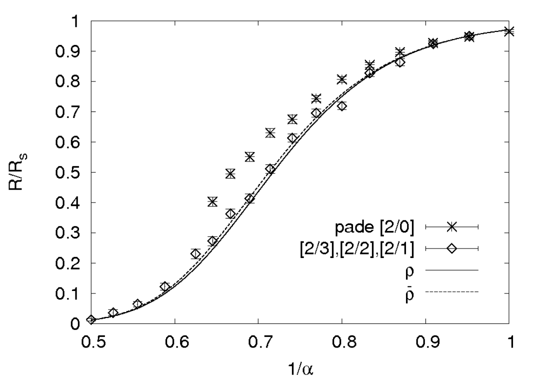
$$E_C = \frac{e^2}{2C} \gg E_J, \quad \alpha = \frac{\pi\hbar}{2e^2}\,R_S^{-1}$$



Exact solution versus MC simulations

Dissipative phase transition

New algorithms: Werner-Troyer (2005) (Data from Lukyanov-Werner (2007))



The figure shows $\frac{R}{R_S}$ as a function of α. Here, the MC data were obtained for $\beta E_C = 1500$, $\frac{E_J}{E_C} = \frac{1}{8}$ and $\Delta\tau E_C = \frac{1}{8}$

Conclusion and outlook

- The Caldeira-Leggett model for a dissipative quantum particle in the titled washboard potential (invented in the 80's) describes a number of important physical problems
  - Point contact current in the fractional quantum Hall effect (Kane-Fisher, Fendley-Ludwig-Saleur (1994)
  - Ultra-small Josephson junctions
- The exact solution obtained as a by-product of purely theoretical studies of integrable structure of two-dimensional quantum field theory. It required state-of-art new advances in the theory: representations of quantum algebras, connection to spectral theory of ODE, etc.
- The same mathematics describes the interference between two one-dimensional Bose condensates (Gritsev et al Nature (2006))

Conclusion and outlook

- There a few dissipative models for "quantum dots" which can be solved exactly, for example
  - Coulumb charging of a low-capacitance metallic island (quantum dot) through a tunnel junction. Ambegaokar-Eckern-Schön dissipative action (1982)

$$S_{diss} \simeq \frac{\alpha}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \Big(\frac{\sin(\phi(t) - \phi(t'))}{(t - t')} \Big)^2$$

  - This problem is described by a "boundary brane" model (Lukyanov-Zamolodchikov (2004))

$$X(0,t)^2 + Y(0,t)^2 = const$$

  - Coqblin-Schrieffer model with generalized fields. Kondo type problem for high spin impurities. (Bazhanov-Lukyanov-Tsvelik (2004)).

If the curtain opens on a play with a rifle on the set, then it certainly be fired by someone by the third act.

A.P.Chekhov